

# The Methods of Chapman-Enskog and Grad and Applications

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# 1 Introduction

We may say that one of the main objectives of the kinetic theory is to describe the macroscopic properties of gases – like pressure, temperature, thermal conductivity, viscosity, diffusion, etc. – from microscopic quantities that are associated with the molecules which compose the gases – like mass, velocity, kinetic energy, internal degrees of freedom and interaction forces between the molecules.

The foundations of the modern kinetic theory of gases were established in 1867 by James Clerk Maxwell (1831-1879) who proposed a general transport equation for arbitrary macroscopic quantities associated with mean values of microscopic quantities. This equation of transport relates the time evolution of a macroscopic quantity with the motion of the molecules, collision between the molecules and action of external forces. Although the theory was valid for any molecular interaction potential, Maxwell could only determine the transport coefficients of shear viscosity, thermal conductivity and diffusion by assuming that the interaction potential was derived from a repulsive force which was inversely proportional to the fifth power of the relative distance between the molecules. Nowadays this type of potential is known as Maxwellian potential. The kinetic theory of gases gained a new impulse in 1872 with the work by Ludwig Eduard Boltzmann (1844-1906), who proposed an integro-differential equation – the Boltzmann equation – which represents the evolution of the velocity distribution function in the phase space spanned by the coordinates and velocities of the molecules. In the Boltzmann equation the temporal change of the distribution function has two terms, one of them is a drift term due to the motion of the molecules while the other one is a collision term related to encounters of the molecules. Based on this equation, Boltzmann proposed the so-called  $\mathcal{H}$  function which decreases with time or remains constant. The identification of this function as the negative of the gas entropy gave a molecular interpretation of the increase of the entropy for irreversible processes. Furthermore, Boltzmann in the same work presented a rigorous deduction of the Maxwellian distribution function.

From the Boltzmann equation one could determine the velocity distribution function hence the transport coefficients of rarefied gases, however this task was not so easy. Indeed, it took almost forty years after the proposition of Boltzmann equation, for David Hilbert (1862-1943) to show how one could get an approximate solution of the integro-differential equation from a power series expansion of a parameter which is proportional to the mean free path. Further advances were due to Sydney Chapman (1888-1970) and David Enskog (1884-1947) who – in the years 1916 and 1917 – calculated independently and by different methods the transport coefficients for gases whose molecules interact according to any kind of spherically symmetric potential function. Another method derived from the Boltzmann equation was proposed in 1949 by Harold Grad (1923-1986) who expanded the distribution function in terms of tensorial Hermite polynomials and introduced balance equations corresponding to higher order moments of the distribution function.

The aim of these notes is to discuss the methods of Chapman-Enskog and Grad with an applications to granular gases.

(a) Cartesian notation for tensors is used with Latin indexes  $i, j, k, \dots$  – which may range from 1 to 3 – denoting the three-dimensional system of spatial coordinates  $x, y, z$ ;

(b) Einstein's summation convention over repeated indexes is used, for example,  $T_{ij}v_j \equiv \sum_{j=1}^3 T_{ij}v_j$ ;

(c) Parentheses around the indexes denote the symmetric part of a tensor, brackets its antisymmetric part while angular parentheses refer to its traceless symmetric part, for example,

$$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}), \quad T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}), \quad T_{\langle ij \rangle} = T_{(ij)} - \frac{1}{3}T_{rr}\delta_{ij}.$$

Above,  $\delta_{ij}$  is Kronecker's symbol and the traceless tensor  $T_{\langle ij \rangle}$  is called tensor deviator.

## 2 Boltzmann and Balance Equations

We consider a monatomic rarefied gas with  $N$  molecules enclosed in a recipient of volume  $V$ , where a molecule is specified as a point in a six-dimensional phase space spanned by its coordinates  $\mathbf{x} = (x_1, x_2, x_3)$  and velocity components  $\mathbf{c} = (c_1, c_2, c_3)$ . The state of a gas is characterized by the one-particle distribution function  $f(\mathbf{x}, \mathbf{c}, t)$  such that  $f(\mathbf{x}, \mathbf{c}, t)d\mathbf{x}d\mathbf{c}$  gives at time  $t$ , the number of molecules in the volume element with position vectors within the range  $\mathbf{x}$  and  $\mathbf{x} + d\mathbf{x}$  and with velocity vectors within the range  $\mathbf{c}$  and  $\mathbf{c} + d\mathbf{c}$ .

An elastic binary collision is characterized by the conservation laws of momentum and kinetic energy, namely,

$$m\mathbf{c} + m\mathbf{c}_1 = m\mathbf{c}' + m\mathbf{c}'_1, \quad \frac{1}{2}mc^2 + \frac{1}{2}mc_1^2 = \frac{1}{2}mc'^2 + \frac{1}{2}mc_1'^2, \quad (1)$$

where  $(\mathbf{c}, \mathbf{c}_1)$  refer to pre-collisional velocities and  $(\mathbf{c}', \mathbf{c}'_1)$  to post-collisional velocities of two molecules. The subindex 1 is used in order to distinguish two identical molecules that participate in the collision. The relative velocity is denoted by  $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}$  and the energy conservation law implies that  $g = g'$ .

The space-time evolution of the one-particle distribution function in the phase space is governed by the Boltzmann equation, which in the absence of external forces, reads

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = \int (f'_1 f' - f_1 f) g b db d\varepsilon d\mathbf{c}_1. \quad (2)$$

We note that it is a non-linear integro-differential equation for  $f(\mathbf{x}, \mathbf{c}, t)$ . The right-hand side of the Boltzmann equation is related to the collisions of the molecules through the product of two distribution functions. The relative motion of the molecules is characterized by the impact parameter  $0 \leq b < \infty$  and by the azimuthal angle  $0 \leq \varepsilon \leq 2\pi$  (see Figure 1). Furthermore, the following abbreviations were introduced  $f' \equiv f(\mathbf{x}, \mathbf{c}', t)$ ,  $f'_1 \equiv f(\mathbf{x}, \mathbf{c}'_1, t)$ ,  $f \equiv f(\mathbf{x}, \mathbf{c}, t)$ ,  $f_1 \equiv f(\mathbf{x}, \mathbf{c}_1, t)$ .

The multiplication of the Boltzmann equation (2) by an arbitrary function  $\psi \equiv \psi(\mathbf{x}, \mathbf{c}, t)$  and the integration of the resulting equation over all values of the velocity components  $\mathbf{c}$  leads to

$$\begin{aligned} \frac{\partial}{\partial t} \int \psi f d\mathbf{c} + \frac{\partial}{\partial x_i} \int \psi c_i f d\mathbf{c} - \int \left[ \frac{\partial \psi}{\partial t} + c_i \frac{\partial \psi}{\partial x_i} \right] f d\mathbf{c} &= \int (\psi' - \psi) f_1 f g b db d\varepsilon d\mathbf{c}_1 d\mathbf{c} \\ &= \frac{1}{4} \int (\psi_1 + \psi - \psi'_1 - \psi') (f'_1 f' - f_1 f) g b db d\varepsilon d\mathbf{c}_1 d\mathbf{c}, \end{aligned} \quad (3)$$

where the two equalities in the right-hand side of the above equation were obtained from the symmetry properties of the collision operator.

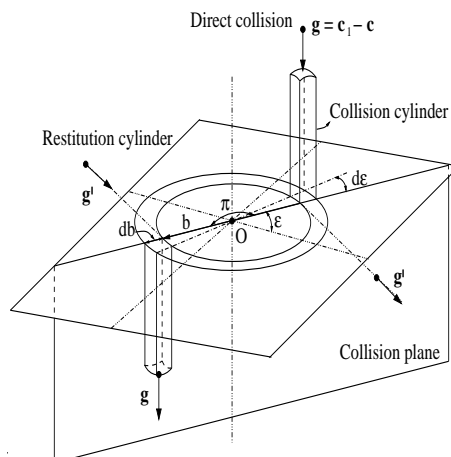


Figure 1: Geometry of a binary collision.

From the analysis of the last equality in the right-hand side of (3) we may infer that it vanishes for any kind of distribution function  $f$  when  $\psi_1 + \psi = \psi'_1 + \psi'$ . A function  $\psi$  which fulfills such condition is called a summational invariant. A summational invariant is given by a linear combination of the conservative quantities: mass, linear momentum and energy, and is expressed by  $\psi(\mathbf{c}) = A + \mathbf{B} \cdot \mathbf{c} + Dc^2$  where  $A$  and  $D$  are two scalar functions and  $\mathbf{B}$  a vectorial function, all of them being independent of  $\mathbf{c}$ .

In kinetic theory a macroscopic state of a gas is characterized by quantities that are defined in terms of the distribution function  $f(\mathbf{x}, \mathbf{c}, t)$ . Firstly, based on the microscopic quantities of the gas molecules as mass  $m$ , linear momentum  $mc_i$  and energy  $mc^2/2$  we define the mass density  $\varrho$ , the momentum density  $\varrho v_i$  and the total energy density  $\varrho u$  of the gas, namely,

$$\varrho = \int m f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}, \quad \varrho v_i = \int m c_i f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}, \quad \varrho u = \int \frac{m c^2}{2} f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}. \quad (4)$$

If we substitute the molecular velocity in (4)<sub>3</sub> by the peculiar velocity  $C_i = c_i - v_i$  – such that  $\int m C_i f d\mathbf{c} = 0$  – we obtain

$$\varrho u = \frac{1}{2}\varrho v^2 + \varrho \varepsilon, \quad \text{where} \quad \varrho \varepsilon = \frac{1}{2} \int m C^2 f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}. \quad (5)$$

Hence, the total energy density of the gas is given by a sum of its kinetic energy density  $\rho v^2/2$  and its internal energy density  $\rho \epsilon$ .

We define the moment of the distribution function of order  $N$  by

$$p_{i_1 i_2 \dots i_N} = \int m C_{i_1} C_{i_2} \dots C_{i_N} f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}, \quad (6)$$

which represents a symmetric tensor of order  $N$  with  $(N+1)(N+2)/2$  distinct components.

The zeroth moment represents the mass density, the first moment vanishes and the second moment, known as the pressure tensor, reads

$$p_{ij} = \int m C_i C_j f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}. \quad (7)$$

We denote by  $p_{\langle ij \rangle}$  the pressure deviator, which represents the traceless part of the pressure tensor. It is given by

$$p_{\langle ij \rangle} = p_{ij} - \frac{1}{3} p_{rr} \delta_{ij}, \quad \text{with} \quad p = \frac{1}{3} p_{rr} = \frac{1}{3} \int m C^2 f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}, \quad (8)$$

representing the hydrostatic pressure of the gas. The internal energy density  $\varrho \varepsilon$ , given by (5)<sub>2</sub> is related to the hydrostatic pressure by  $p = 2\varrho \varepsilon / 3$ . The equation of state of an ideal gas is given by  $p = nkT$  where  $k$ ,  $n = \varrho / m$  and  $T$  denote the Boltzmann constant, the particle number density and the temperature of the gas, respectively. Hence, we can obtain the following expression for the temperature of a monatomic gas written in terms of the distribution function:

$$T = \frac{p}{nk} = \frac{2}{3} \frac{m}{k} \varepsilon = \frac{1}{3nk} \int m C^2 f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}. \quad (9)$$

The heat flux vector is defined as the contracted third order moment

$$q_i = \frac{1}{2} p_{jji} = \frac{1}{2} \int m C^2 C_i f(\mathbf{x}, \mathbf{c}, t) d\mathbf{c}, \quad (10)$$

and the moments of higher order do not have specific proper names.

We obtain the balance equations for the moments of the distribution function from the transfer equation (3) by choosing  $\psi(\mathbf{x}, \mathbf{c}, t)$  equal to:

(i) *Balance of mass density* ( $\psi = m$ ):

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \quad (11)$$

(ii) *Balance of momentum density* ( $\psi = m c_i$ ):

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\varrho v_i v_j + p_{ij}) = 0, \quad (12)$$

(iii) *Balance of total energy density* ( $\psi = m c^2 / 2$ ):

$$\frac{\partial}{\partial t} \left[ \varrho \left( \varepsilon + \frac{1}{2} v^2 \right) \right] + \frac{\partial}{\partial x_i} \left[ \varrho \left( \varepsilon + \frac{1}{2} v^2 \right) v_i + q_i + p_{ij} v_j \right] = 0, \quad (13)$$

(iv) *Balance of Nth order moment* ( $\psi = m C_{i_1} C_{i_2} \dots C_{i_N}$ ):

$$\begin{aligned} \frac{\partial p_{i_1 i_2 \dots i_N}}{\partial t} + \frac{\partial}{\partial x_k} (p_{i_1 i_2 \dots i_N k} + p_{i_1 i_2 \dots i_N} v_k) - \frac{N}{\varrho} p_{(i_1 i_2 \dots i_{N-1}} \frac{\partial p_{i_N)k}}{\partial x_k} \\ + N p_{k(i_1 i_2 \dots i_{N-1}} \frac{\partial v_{i_N)}}{\partial x_k} = P_{i_1 i_2 \dots i_N}. \end{aligned} \quad (14)$$

The above equation was obtained by eliminating the time derivative of the hydrodynamic velocity  $v_i$  by the use of (12). The parenthesis around the indexes indicate a sum over all  $N!$  permutations of these indexes divided by  $N!$ . Furthermore, the production term due to the molecular collisions  $P_{i_1 i_2 \dots i_N}$  reads

$$P_{i_1 i_2 \dots i_N} = \int m (C'_{i_1} C'_{i_2} \dots C'_{i_N} - C_{i_1} C_{i_2} \dots C_{i_N}) f f_1 g b db d\varepsilon d\mathbf{c} d\mathbf{c}_1. \quad (15)$$

If we subtract from the balance equation of the total energy density (13) the momentum balance equation (12) multiplied by the bulk velocity  $v_i$ , we obtain the balance equation for the internal energy density, namely,

$$\frac{\partial \varrho \varepsilon}{\partial t} + \frac{\partial}{\partial x_i} (\varrho \varepsilon v_i + q_i) + p_{ij} \frac{\partial v_i}{\partial x_j} = 0. \quad (16)$$

### 3 Chapman-Enskog Method

First let us introduce the dimensionless variables

$$t^* = t/t_c, \quad \mathbf{x}^* = \mathbf{x}/L_c, \quad \mathbf{c}^* = \mathbf{c}/\bar{c}, \quad \mathbf{g}^* = \mathbf{g}/\bar{g}, \quad b^* = b/d, \quad \varepsilon^* = \varepsilon/\pi, \quad f^* = f/f_c, \quad (17)$$

where  $d$  denote the molecular diameter,  $\bar{c} = \sqrt{8kT/m\pi}$  the mean thermal velocity, and  $\bar{g} = \sqrt{2}\bar{c}$  the mean relative velocity. Furthermore, the characteristic value for the distribution function  $f_c$  is equal to  $n/\bar{c}^3$  where  $n$  represents the particle number density of the gas and  $t_c$  and  $L_c$  are a representative time and length that characterize the fluid flow. By taking into account the dimensionless variables (17), the Boltzmann equation (2) can be written as

$$Sr \frac{\partial f^*}{\partial t^*} + c_i^* \frac{\partial f^*}{\partial x_i^*} = \frac{1}{Kn} \int (f_1^* f^* - f_1^* f^*) g^* b^* db^* d\varepsilon^* dc_1^*. \quad (18)$$

In the above equation we have introduced the Strouhal number  $Sr$  and the Knudsen number  $Kn$  which are defined by

$$Sr = \frac{L_c}{\bar{c} t_c}, \quad Kn = \frac{l}{L_c}, \quad (19)$$

where  $l = 1/\sqrt{2}\pi d^2 n$  is the mean free path.

The Knudsen number is related to the degree of rarefaction of a gas. When  $Kn \ll 1$  the molecular collisions are very important, the distribution function is determined by the collision term of the Boltzmann equation and the gas is described by a continuum regime. Otherwise, when  $Kn \gg 1$  the molecular collisions become negligible, the distribution function is determined via a collisionless Boltzmann equation and the regime of the gas is known as free molecular flow.

In the method of Chapman-Enskog the distribution function  $f$  is expanded as

$$f = f^{(0)} + \Lambda f^{(1)} + \Lambda^2 f^{(2)} + \dots = \sum_{r=0}^{\infty} \Lambda^r f^{(r)}, \quad (20)$$

i.e., in power series of a parameter  $\Lambda$  which is of order of the Knudsen number. The distribution functions  $f^{(0)}$ ,  $f^{(1)}$  and  $f^{(2)}$  represent the first, second and third approximation to the distribution function, and so on. The parameter  $\Lambda$  can be set later equal to unity, so that the proper dimensions of the Boltzmann equation are restored. Moreover, the approximations must satisfy the constraints

$$\int \psi f^{(r)} d\mathbf{c} = 0, \quad \forall r \geq 1, \quad (21)$$



where  $\psi$  represents the summational invariants  $m$ ,  $mC_i$  (or  $mC_i$ ) and  $mc^2/2$  (or  $mC^2/2$ ).

If we insert the series expansion of the distribution function (20) into the definitions of the pressure tensor and of the heat flux vector we obtain

$$p_{ij} = \delta_{ij} \frac{1}{3} \int mC^2 f^{(0)} d\mathbf{c} + \int mC_{\langle i} C_{j \rangle} \sum_{r=1}^{\infty} \Lambda^r f^{(r)} d\mathbf{c} = p\delta_{ij} + \sum_{r=1}^{\infty} \Lambda^r p_{\langle ij \rangle}^{(r)}, \quad (22)$$

$$q_i = \int \frac{m}{2} C^2 C_i \sum_{r=1}^{\infty} \Lambda^r f^{(r)} d\mathbf{c} = \sum_{r=1}^{\infty} \Lambda^r q_i^{(r)}. \quad (23)$$

The parameter  $\Lambda$  is introduced into the collision term of the Boltzmann equation (2), yielding

$$\mathcal{D}f + C_i \frac{\partial f}{\partial x_i} = \frac{1}{\Lambda} \mathcal{Q}(f, f), \quad (24)$$

where  $\mathcal{D} = \partial/\partial t + v_i \partial/\partial x_i$  denotes the material time derivative. In (24) the collision term was expressed in terms of an integral for a bilinear quantity

$$\mathcal{Q}(F, G) = \frac{1}{2} \int (F'_1 G' + F' G'_1 - F_1 G - F G_1) g b db d\varepsilon dc_1. \quad (25)$$

The material time derivative, like the distribution function, is also expanded in power series as

$$\mathcal{D} = \mathcal{D}_0 + \Lambda \mathcal{D}_1 + \Lambda^2 \mathcal{D}_2 + \cdots = \sum_{r=1}^{\infty} \Lambda^r \mathcal{D}_r, \quad (26)$$

and the insertion of the expansions (22), (23) and (26) into (11), (12) and (16) lead to the following decomposition of the balance equations

$$\mathcal{D}_0 \varrho + \varrho \frac{\partial v_i}{\partial x_i} = 0, \quad \mathcal{D}_r \varrho = 0 \quad (\forall r \geq 1), \quad (27)$$

$$\varrho \mathcal{D}_0 v_i + \frac{\partial p}{\partial x_i} = 0, \quad \varrho \mathcal{D}_r v_i + \frac{\partial p_{\langle ij \rangle}^{(r)}}{\partial x_j} = 0 \quad (\forall r \geq 1), \quad (28)$$

$$\frac{3}{2} nk \mathcal{D}_0 T + p \frac{\partial v_i}{\partial x_i} = 0, \quad \frac{3}{2} nk \mathcal{D}_r T + \frac{\partial q_i^{(r)}}{\partial x_i} + p_{\langle ij \rangle}^{(r)} \frac{\partial v_i}{\partial x_j} = 0 \quad (\forall r \geq 1). \quad (29)$$

We can now obtain the integral equations for the approximations of the distribution function by the inserting the expansions (20) and (26) into the Boltzmann equation (24) and equating equal powers of  $\Lambda$ . Hence, it follows

$$\mathcal{Q}(f^{(0)}, f^{(0)}) = 0, \quad (30)$$

$$2\mathcal{Q}(f^{(0)}, f^{(1)}) = \mathcal{D}_0 f^{(0)} + C_i \frac{\partial f^{(0)}}{\partial x_i}, \quad (31)$$

$$2\mathcal{Q}(f^{(0)}, f^{(2)}) + \mathcal{Q}(f^{(1)}, f^{(1)}) = \mathcal{D}_0 f^{(1)} + \mathcal{D}_1 f^{(0)} + C_i \frac{\partial f^{(1)}}{\partial x_i}, \quad (32)$$

and so on. The above equations represent the three first integral equations for  $f^{(0)}$ ,  $f^{(1)}$  and  $f^{(2)}$ , respectively.

For the determination of the first approximation to the distribution function  $f^{(0)}$  from the integral equation (30) we note that its solution is given by  $f'^{(0)} f_1^{(0)} = f^{(0)} f_1^{(0)}$  or  $\ln f'^{(0)} + \ln f_1^{(0)} = \ln f^{(0)} + \ln f_1^{(0)}$ . Hence,  $\ln f^{(0)}$  is a summation invariant so that it must be represented by  $\ln f^{(0)} = A + \mathbf{B} \cdot \mathbf{C} + DC^2$ . If we insert this representation for  $f^{(0)}$  into the definitions of the mass density (4)<sub>1</sub>, hydrodynamic velocity (4)<sub>2</sub> and temperature (9) we obtain the Maxwellian distribution function, namely,

$$f^{(0)} = n \left( \frac{\beta}{\pi} \right)^{\frac{3}{2}} e^{-\beta C^2} \quad \text{where} \quad \beta = \frac{m}{2kT}. \quad (33)$$

For the determination of the second approximation  $f^{(1)}$  we write  $f^{(1)} = f^{(0)} \phi$  and the integral equation (31) reduces to

$$\begin{aligned} 2\mathcal{Q}(f^{(0)}, f^{(0)}\phi) &= f^{(0)} \left\{ \frac{1}{T} \left( \beta C^2 - \frac{5}{2} \right) C_i \frac{\partial T}{\partial x_i} + 2\beta C_i C_j \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} \right\} \\ &= \mathcal{I}[\phi] = \int f^{(0)} f_1^{(0)} (\phi'_1 + \phi' - \phi_1 - \phi) g b db d\epsilon d\mathbf{c}_1, \end{aligned} \quad (34)$$

by eliminating the material time derivatives through the use of (27)<sub>1</sub>, (28)<sub>1</sub> and (29)<sub>1</sub>.

The general solution of the integral equation (34) is given by a sum of the solution of the homogeneous integral equation plus a particular solution of it, i.e.,  $\phi = \phi_h + \phi_p$ . The solution of the homogeneous integral equation  $\mathcal{I}[\phi_h] = 0$  is a summational invariant  $\phi_h = \alpha_1 + \alpha_r^2 C_r + \alpha_3 C^2$ , where  $\alpha_1$  and  $\alpha_3$  are scalar functions, while  $\alpha_r^2$  is a vector function, all of them do not depend on the peculiar velocity  $\mathbf{C}$ .

The particular solution of the integral equation is expressed as a linear combination of the temperature gradient and of the velocity gradient deviator, namely,

$$\phi_p = -\frac{A_i}{T} \frac{\partial T}{\partial x_i} - 2\beta B_{ij} \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}. \quad (35)$$

where  $A_i$  and  $B_{ij}$  denote a vector function and a tensor function of  $(\mathbf{C}, \varrho, T)$ , respectively. It is easy to show that the representations of the vector and tensor functions in terms of the peculiar velocity  $\mathbf{C}$  are given by  $A_i = \mathcal{A}^* C_i$  and  $B_{ij} = \mathcal{B} C_i C_j$ , where the scalar functions  $\mathcal{A}^*$  and  $\mathcal{B}$  depend only on  $(C^2, \varrho, T)$ .

Hence, the solution of the non-homogeneous integral equation (34) is given by

$$\phi = -\mathcal{A}^* \frac{C_i}{T} \frac{\partial T}{\partial x_i} - 2\beta \mathcal{B} C_i C_j \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} + \alpha_1 + \alpha_r^2 C_r + \alpha_3 C^2. \quad (36)$$

The second approximation  $f^{(1)} = f^{(0)} \phi$  must satisfy the constraints (21) which imply the following restrictions

$$\alpha_1 + \frac{3kT}{m} \alpha_3 = 0, \quad \alpha_1 + \frac{5kT}{m} \alpha_3 = 0, \quad \int m C^2 \left[ -\frac{\mathcal{A}^*}{T} \frac{\partial T}{\partial x_j} + \alpha_j^2 \right] f^{(0)} d\mathbf{c} = 0. \quad (37)$$

From the two first equations we may infer that  $\alpha_1 = \alpha_3 = 0$ , whereas the last equation implies that  $\alpha_j^2$  must be proportional to  $\partial T / \partial x_j$ . By writing the proportionality factor of  $\alpha_j^2$  as  $\alpha/T$ , the constraint (37)<sub>3</sub> reduces to

$$\int C^2 \mathcal{A} f^{(0)} d\mathbf{c} = 0, \quad (38)$$

with  $\mathcal{A} = \mathcal{A}^* - \alpha$  being a new scalar function of  $(C^2, \varrho, T)$ .

Hence, from the above results we may write the distribution function (20) – up to the second approximation and with  $\Lambda = 1$  – as

$$f = f^{(0)}(1 + \phi) = f^{(0)} \left\{ 1 - \frac{\mathcal{A}}{T} C_i \frac{\partial T}{\partial x_i} - 2\beta \mathcal{B} C_i C_j \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} \right\}. \quad (39)$$

Now the scalar coefficients  $\mathcal{A}$  and  $\mathcal{B}$  in the distribution function (39) can be obtained as solutions of the integral equations that follows from (34), namely,

$$f^{(0)} \left[ \beta C^2 - \frac{5}{2} \right] C_i = -\mathcal{I}[\mathcal{A} C_i], \quad f^{(0)} C_{\langle i} C_{j \rangle} = -\mathcal{I}[\mathcal{B} C_{\langle i} C_{j \rangle}]. \quad (40)$$

In order to solve the integral equations (40) the scalar coefficients  $\mathcal{A}(C^2, \varrho, T)$  and  $\mathcal{B}(C^2, \varrho, T)$  are expanded in series of Sonine polynomials. The Sonine polynomials of order  $n$  and index  $(l + 1/2)$  in the variable  $\beta C^2 = mC^2/2kT$  are defined by

$$S_{l+1/2}^{(n)}(\beta C^2) = \sum_{k=0}^n \frac{\Gamma(n + l + 3/2)}{k!(n - k)!\Gamma(k + l + 3/2)} (-\beta C^2)^k, \quad (41)$$

and obey the orthogonality conditions

$$\int_0^\infty e^{-\beta C^2} C^{2l+2} S_{l+1/2}^{(n)}(\beta C^2) S_{l+1/2}^{(p)}(\beta C^2) \beta^{l+3/2} dC = \frac{1}{2} \frac{\Gamma(n + l + 3/2)}{n!} \delta_{np}, \quad (42)$$

where  $\Gamma(n)$  denotes the gamma function. The first three Sonine polynomials read

$$S_{l+1/2}^{(0)}(\beta C^2) = 1, \quad S_{l+1/2}^{(1)}(\beta C^2) = l + \frac{3}{2} - \beta C^2, \quad (43)$$

$$S_{l+1/2}^{(2)}(\beta C^2) = \frac{1}{2} \left( l + \frac{5}{2} \right) \left( l + \frac{3}{2} \right) - \left( l + \frac{5}{2} \right) \beta C^2 + \frac{1}{2} \beta^2 C^4. \quad (44)$$

The integral equations (40) in terms of the Sonine polynomials become

$$f^{(0)} S_{3/2}^{(1)}(\beta C^2) C_i = \mathcal{I}[\mathcal{A} C_i], \quad f^{(0)} S_{5/2}^{(0)}(\beta C^2) C_{\langle i} C_{j \rangle} = -\mathcal{I}[\mathcal{B} C_{\langle i} C_{j \rangle}]. \quad (45)$$

Moreover, the expansions of the scalar coefficients  $\mathcal{A}(C^2, \varrho, T)$  and  $\mathcal{B}(C^2, \varrho, T)$  in series of Sonine polynomials are written as

$$\mathcal{A}(C^2, \varrho, T) = - \sum_{r=0}^{\infty} a^{(r)} S_{3/2}^{(r)}(\beta C^2), \quad \mathcal{B}(C^2, \varrho, T) = \sum_{r=0}^{\infty} b^{(r)} S_{5/2}^{(r)}(\beta C^2). \quad (46)$$

In the above equations, the scalar coefficients  $a^{(r)}$  and  $b^{(r)}$  are only functions of  $(\varrho, T)$ , and from this point on, the dependence of the Sonine polynomials in the variable  $\beta C^2$  is omitted.

The coefficient  $\mathcal{A}$  must satisfy the constraint (38), hence by using its representation (46)<sub>1</sub>, we get

$$0 = \int C^2 f^{(0)} \sum_{r=0}^{\infty} a^{(r)} S_{3/2}^{(r)} d\mathbf{c} = 4\pi n \left( \frac{\beta}{\pi} \right)^{\frac{3}{2}} \sum_{r=0}^{\infty} a^{(r)} \int_0^\infty S_{3/2}^{(0)} S_{3/2}^{(r)} C^4 e^{-\beta C^2} dC = \frac{3}{2} \frac{n}{\beta} a^{(0)}, \quad (47)$$

due to (42) and (43)<sub>1</sub>. From (47) it follows that  $a^{(0)} = 0$  and the scalar coefficient  $\mathcal{A}$  reduces to

$$\mathcal{A}(C^2, \varrho, T) = - \sum_{r=1}^{\infty} a^{(r)} S_{3/2}^{(r)}. \quad (48)$$

Now the insertion of the expressions (48) and (46)<sub>2</sub> into their respective integral equations (45), yields

$$f^{(0)} S_{3/2}^{(1)} C_i = - \sum_{r=1}^{\infty} a^{(r)} \mathcal{I} \left[ S_{3/2}^{(r)} C_i \right], \quad f^{(0)} S_{5/2}^{(0)} C_{\langle i} C_{j \rangle} = - \sum_{r=0}^{\infty} b^{(r)} \mathcal{I} \left[ S_{5/2}^{(r)} C_{\langle i} C_{j \rangle} \right]. \quad (49)$$

In order to determine the coefficients  $a^{(r)}$  e  $b^{(r)}$  from (49) we proceed as follows. First we multiply the integral equation (49)<sub>1</sub> by  $\beta S_{3/2}^{(s)} C_i$  and integrate the resulting equation over all values of  $\mathbf{C}$ . By using (42) we obtain

$$\frac{15}{4} \frac{n}{\beta^2} \delta^{(1,r)} = \sum_{s=1}^{\infty} \alpha^{(r,s)} a^{(s)}, \quad \text{where} \quad \alpha^{(r,s)} = - \frac{1}{\beta} \int S_{3/2}^{(r)} C_i \mathcal{I} [S_{3/2}^{(s)} C_i] d\mathbf{C}. \quad (50)$$

In the above equation  $\delta^{(1,r)}$  represents Kronecker's symbol. Following the same methodology, we multiply the integral equation (49)<sub>2</sub> by  $\beta^2 S_{5/2}^{(s)} C_{\langle i} C_{j \rangle}$ , integrate the resulting equation and get

$$\frac{5}{2} \frac{n}{\beta^2} \delta^{(0,r)} = \sum_{s=0}^{\infty} \beta^{(r,s)} b^{(s)}, \quad \text{where} \quad \beta^{(r,s)} = - \int S_{5/2}^{(r)} C_{\langle i} C_{j \rangle} \mathcal{I} [S_{5/2}^{(s)} C_{\langle i} C_{j \rangle}] d\mathbf{C}. \quad (51)$$

The integrals in (50)<sub>2</sub> and (51)<sub>2</sub> depend on the molecular interaction potential and on the fields  $(\varrho, T)$ . Moreover, from the condition  $\int \varphi \mathcal{I}[\phi] d\mathbf{c} = \int \phi \mathcal{I}[\varphi] d\mathbf{c}$  – which is valid for all arbitrary functions  $\phi(\mathbf{x}, \mathbf{c}, t)$  and  $\varphi(\mathbf{x}, \mathbf{c}, t)$  – we may infer that

$$\alpha^{(r,s)} = \alpha^{(s,r)}, \quad \beta^{(r,s)} = \beta^{(s,r)} \quad \text{and} \quad \alpha^{(r,0)} = \alpha^{(0,r)} = 0. \quad (52)$$

We conclude that (50)<sub>1</sub> and (51)<sub>1</sub> represent an infinite system of algebraic equations for the coefficients  $a^{(n)}$  and  $b^{(n)}$ , respectively.

To obtain the constitutive equations for the pressure tensor and for the heat flux vector we proceed as follows. First, we insert the non-equilibrium distribution function (39) together with the representations of  $\mathcal{A}$  and  $\mathcal{B}$  – given by (48) and (46)<sub>2</sub>, respectively – into the definitions of the pressure tensor and heat flux vector and get

$$p_{ij} = \int m C_i C_j f d\mathbf{c} = p \delta_{ij} - \int 2m \beta C_i C_j S_{5/2}^{(0)} f^{(0)} \sum_{r=0}^{\infty} b^{(r)} S_{5/2}^{(r)} C_{\langle p} C_{q \rangle} \frac{\partial v_p}{\partial x_q} d\mathbf{c}, \quad (53)$$

$$q_i = \int \frac{m}{2} C^2 C_i f d\mathbf{c} = \int \frac{m}{2T\beta} \left[ \frac{5}{2} S_{3/2}^{(0)} - S_{3/2}^{(1)} \right] C_i f^{(0)} \sum_{r=1}^{\infty} a^{(r)} S_{3/2}^{(r)} C_p \frac{\partial T}{\partial x_p} d\mathbf{c}. \quad (54)$$

In (54) we have used the relationship  $\beta C^2 = \left[ \frac{5}{2} S_{3/2}^{(0)} - S_{3/2}^{(1)} \right]$  which follows from (43)<sub>2</sub>. Next, the substitution of the Maxwellian distribution function (33) into (53) and (54) and

the subsequent integration of the resulting equations over all values of  $\mathbf{C}$ , yield

$$p_{ij} = p \delta_{ij} - 2\mu \frac{\partial v_{\langle i}}{\partial x_{j\rangle}}, \quad \text{where} \quad \mu = \frac{\varrho}{2\beta} b^{(0)}, \quad (55)$$

$$q_i = -\lambda \frac{\partial T}{\partial x_i}, \quad \text{where} \quad \lambda = \frac{5k}{4m} \frac{\varrho}{\beta} a^{(1)}, \quad (56)$$

thanks to the orthogonality condition of the Sonine polynomials (42). Equations (55)<sub>1</sub> and (56)<sub>1</sub> are the mathematical expressions for the laws of Navier-Stokes and Fourier, with  $\mu$  and  $\lambda$  representing the coefficients of shear viscosity and thermal conductivity, respectively.

We note from (55)<sub>2</sub> and (56)<sub>2</sub> that the transport coefficients of shear viscosity and thermal conductivity are functions only of the coefficients  $a^{(1)}$  and  $b^{(0)}$ , respectively. We can obtain these coefficients from the system of equations (50)<sub>1</sub> and (51)<sub>1</sub>, which can be written as

$$a^{(1)} = \lim_{n \rightarrow \infty} \frac{\mathcal{A}'_{nn}}{\mathcal{A}_{nn}}, \quad b^{(0)} = \lim_{n \rightarrow \infty} \frac{\mathcal{B}'_{nn}}{\mathcal{B}_{nn}}, \quad (57)$$

where

$$\mathcal{A}'_{nn} = \det \begin{vmatrix} \frac{15}{4} \frac{n}{\beta^2} & \alpha^{(1,2)} & \dots & \alpha^{(1,n)} \\ 0 & \alpha^{(2,2)} & \dots & \alpha^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha^{(n,2)} & \dots & \alpha^{(n,n)} \end{vmatrix}, \quad \mathcal{A}_{nn} = \det \begin{vmatrix} \alpha^{(1,1)} & \dots & \alpha^{(1,n)} \\ \vdots & \ddots & \vdots \\ \alpha^{(n,1)} & \dots & \alpha^{(n,n)} \end{vmatrix}, \quad (58)$$

$$\mathcal{B}'_{nn} = \det \begin{vmatrix} \frac{5}{2} \frac{n}{\beta^2} & \beta^{(0,1)} & \dots & \beta^{(0,n)} \\ 0 & \beta^{(1,1)} & \dots & \beta^{(1,n)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta^{(n,1)} & \dots & \beta^{(n,n)} \end{vmatrix}, \quad \mathcal{B}_{nn} = \det \begin{vmatrix} \beta^{(0,0)} & \dots & \beta^{(0,n)} \\ \vdots & \ddots & \vdots \\ \beta^{(n,0)} & \dots & \beta^{(n,n)} \end{vmatrix}. \quad (59)$$

The coefficients  $a^{(1)}$  and  $b^{(0)}$  follow from (58) and (59) through a method of successive approximations, where the approximation of order  $p$  for the coefficients  $a^{(1)}$  and  $b^{(0)}$  – denoted by  $[a^{(1)}]_p$  and  $[b^{(0)}]_p$  – is given by the sub-determinant of order  $p$ . The first two approximations for the coefficients  $a^{(1)}$  e  $b^{(0)}$  are

$$[a^{(1)}]_1 = \frac{15}{4} \frac{n}{\beta^2} \frac{1}{\alpha^{(1,1)}}, \quad [b^{(0)}]_1 = \frac{5}{2} \frac{n}{\beta^2} \frac{1}{\beta^{(0,0)}}, \quad (60)$$

$$[a^{(1)}]_2 = \frac{15}{4} \frac{n}{\beta^2} \frac{\alpha^{(2,2)}}{\alpha^{(1,1)}\alpha^{(2,2)} - \alpha^{(1,2)^2}}, \quad [b^{(0)}]_2 = \frac{5}{2} \frac{n}{\beta^2} \frac{\beta^{(1,1)}}{\beta^{(0,0)}\beta^{(1,1)} - \beta^{(0,1)^2}}. \quad (61)$$

The integrals  $\alpha^{(r,s)}$  and  $\beta^{(r,s)}$  are given in terms of the collision integrals

$$\Omega^{(l,r)} = \int_0^\infty \int_0^\infty e^{-\gamma^2} \gamma^{2r+3} (1 - \cos^l \chi) b db d\gamma, \quad (62)$$

where  $\chi = \arccos(\mathbf{g}' \cdot \mathbf{g}/g^2)$  is the scattering angle and  $\gamma = \sqrt{\beta/2} g$  represents a dimensionless relative velocity.

The determination of the integrals  $\alpha^{(r,s)}$  and  $\beta^{(r,s)}$  is not an easy task. As an example, let us compute the integral  $\beta^{(0,0)}$ . For its calculation a change of the integration variables

is performed by replacing the velocities  $\mathbf{C}$  and  $\mathbf{C}_1$  by the relative velocity  $\mathbf{g} = \mathbf{C}_1 - \mathbf{C}$  and by the center of mass velocity  $\mathbf{G} = (\mathbf{C}_1 + \mathbf{C})/2$ . The linear momentum conservation law implies that  $\mathbf{G}' = \mathbf{G}$  and the Jacobian of the transformation of the volume elements has modulus equal to one so that  $d\mathbf{C}d\mathbf{C}_1 = d\mathbf{g}d\mathbf{G}$ . In terms of the new variables the integral reads

$$\begin{aligned} \beta^{(0,0)} &= \int C_{\langle i} C_{j \rangle} \mathcal{I}[C_{\langle i} C_{j \rangle}] d\mathbf{c} = \frac{n^2}{2} \left( \frac{\beta}{\pi} \right)^3 \int e^{-2\beta G^2} e^{-\frac{\beta}{2} g^2} \\ &\times \left( G_{\langle i} G_{j \rangle} - G_{\langle i} g_{j \rangle} + \frac{1}{4} g_{\langle i} g_{j \rangle} \right) (g'_{\langle i} g'_{j \rangle} - g_{\langle i} g_{j \rangle}) g b db d\varepsilon d\mathbf{g} d\mathbf{G}. \end{aligned} \quad (63)$$

The integration of (63) with respect to  $\mathbf{G}$  and  $\varepsilon$  leads to

$$\beta^{(0,0)} = \frac{\pi n^2}{4} \left( \frac{\beta}{2\pi} \right)^{\frac{3}{2}} \int e^{-\frac{\beta}{2} g^2} (\cos^2 \chi - 1) g^5 b db d\mathbf{g}, \quad (64)$$

If we introduce the dimensionless relative velocity  $\gamma = \sqrt{\beta/2} g$ , the integration of the above equation with respect to the angles of  $\mathbf{g}$  leads to

$$\beta^{(0,0)} = -\sqrt{\pi} n^2 \left( \frac{2}{\beta} \right)^{\frac{5}{2}} \Omega^{(2,2)}, \quad (65)$$

where  $\Omega^{(2,2)}$  denotes the collision integral defined by (62).

By introducing the integrals  $\alpha_{\star}^{(r,s)} = \frac{\beta^{\frac{5}{2}}}{n^2 \sqrt{2\pi}} \alpha^{(r,s)}$ , and  $\beta_{\star}^{(r,s)} = \frac{\beta^{\frac{5}{2}}}{n^2 \sqrt{2\pi}} \beta^{(r,s)}$ , we can obtain in the same way the following expressions in terms of the collision integrals  $\Omega^{(l,r)}$

$$\alpha_{\star}^{(1,1)} = 4\Omega^{(2,2)}, \quad \alpha_{\star}^{(1,2)} = 7\Omega^{(2,2)} - 2\Omega^{(2,3)}, \quad \alpha_{\star}^{(2,2)} = \frac{77}{4}\Omega^{(2,2)} - 7\Omega^{(2,3)} + \Omega^{(2,4)}, \quad (66)$$

$$\beta_{\star}^{(0,0)} = \alpha_{\star}^{(1,1)}, \quad \beta_{\star}^{(0,1)} = \alpha_{\star}^{(1,2)}, \quad \beta_{\star}^{(1,1)} = \frac{301}{12}\Omega^{(2,2)} - 7\Omega^{(2,3)} + \Omega^{(2,4)} = \alpha_{\star}^{(2,2)} + \frac{35}{24}\alpha_{\star}^{(1,1)}. \quad (67)$$

As was pointed out the coefficients  $b^{(0)}$  and  $a^{(1)}$  are determined through a method of successive approximations, so are the transport coefficients  $\mu$  and  $\lambda$ . Hence, it follows from (60), (61), (55)<sub>2</sub> and (56)<sub>2</sub> that the two first approximations to the coefficients of shear viscosity and thermal conductivity read

$$[\mu]_1 = \frac{5}{4} \sqrt{\frac{mkT}{\pi}} \frac{1}{\beta_{\star}^{(0,0)}}, \quad [\lambda]_1 = \frac{75k}{16m} \sqrt{\frac{mkT}{\pi}} \frac{1}{\alpha_{\star}^{(1,1)}}, \quad (68)$$

$$[\mu]_2 = \frac{5}{4} \sqrt{\frac{mkT}{\pi}} \frac{\beta_{\star}^{(1,1)}}{\beta_{\star}^{(0,0)} \beta_{\star}^{(1,1)} - \beta_{\star}^{(1,2)^2}}, \quad [\lambda]_2 = \frac{75k}{16m} \sqrt{\frac{mkT}{\pi}} \frac{\alpha_{\star}^{(2,2)}}{\alpha_{\star}^{(1,1)} \alpha_{\star}^{(2,2)} - \alpha_{\star}^{(1,2)^2}}. \quad (69)$$

Now from (68) and (69) we can build the ratios

$$\frac{[\lambda]_1}{[\mu]_1} = \frac{5}{2} c_v, \quad \frac{[\lambda]_2}{[\lambda]_1} = \left( 1 - \frac{\alpha_{\star}^{(1,2)^2}}{\alpha_{\star}^{(1,1)} \alpha_{\star}^{(2,2)}} \right)^{-1} > \left( 1 - \frac{\beta_{\star}^{(0,1)^2}}{\beta_{\star}^{(0,0)} \beta_{\star}^{(1,1)}} \right)^{-1} = \frac{[\mu]_2}{[\mu]_1}. \quad (70)$$

We may infer from (70)<sub>1</sub> that the ratio of the first approximations to the coefficients of thermal conductivity and shear viscosity of monatomic ideal gases is equal to  $5c_v/2$  for all types of spherically symmetrical molecular interaction potentials, where  $c_v = 3k/2m$  is the specific heat at constant volume. Furthermore, from the relationship (70)<sub>2</sub> it follows that  $[\lambda]_2/[\mu]_2 > [\lambda]_1/[\mu]_1 = 5c_v/2$ , i.e., the ratio of the second approximations is larger than the ratio of the first ones.

For a hard-sphere potential the impact parameter is given by  $b = d \cos(\chi/2)$  with  $0 \leq \chi \leq \pi$ , so that the collision integral  $\Omega^{(2,r)}$  becomes

$$\Omega^{(2,r)} = \int_0^\infty e^{-\gamma^2} \gamma^{2r+3} \frac{1}{3} d^2 d\gamma = \frac{d^2}{6} \Gamma\left(\frac{2r+4}{2}\right). \quad (71)$$

Hence, it follows from (66) and (67) that

$$\alpha_\star^{(1,1)} = \beta_\star^{(0,0)} = 4d^2, \quad \alpha_\star^{(1,2)} = \beta_\star^{(0,1)} = -d^2, \quad \alpha_\star^{(2,2)} = \frac{45}{4}d^2, \quad \beta_\star^{(1,1)} = \frac{205}{12}d^2, \quad (72)$$

and the two first approximations (68) and (69) to the coefficients of shear viscosity and thermal conductivity for the hard-sphere potential read

$$[\mu]_1 = \frac{5}{16d^2} \sqrt{\frac{mkT}{\pi}}, \quad \frac{[\lambda]_1}{[\mu]_1} = \frac{5}{2}c_v, \quad \frac{[\mu]_2}{[\mu]_1} = \frac{205}{202} \approx 1.014851, \quad \frac{[\lambda]_2}{[\lambda]_1} = \frac{45}{44} \approx 1.022727. \quad (73)$$

We note that the second approximations to the transport coefficients is a small correction to the first ones, so is the ratio  $[\lambda]_2/[\mu]_2 \approx 1.007761[\lambda]_1/[\mu]_1$ .

The fifth approximations to the transport coefficients for hard-sphere potential are

$$\frac{[\mu]_5}{[\mu]_1} \approx 1.016027, \quad \frac{[\lambda]_5}{[\lambda]_1} \approx 1.025197.$$

## 4 Application of Chapman-Enskog Method to Granular Gases

The mechanical energy of a gas is conserved when its molecules undergo elastic collisions and, in this case, the gas relaxes towards an equilibrium state characterized by a Maxwellian distribution function. However, the inelastic collisions of the gas molecules transform the translational kinetic energy into heat and the mechanical energy lost implies a temperature decay of the gas. In the literature gases whose molecules undergo inelastic collisions are known as granular gases. The main premisses of the kinetic theory of granular gases are: **(a)** only binary collisions of hard spherical molecules are taken into account and **(b)** the energy lost from inelastic collisions is small.

Let us consider the encounter of two identical molecules of mass  $m$  diameter  $d$  pre-collisional velocities  $(\mathbf{c}, \mathbf{c}_1)$  and post-collisional velocities  $(\mathbf{c}', \mathbf{c}'_1)$ . The momentum conservation law  $m\mathbf{c} + m\mathbf{c}_1 = m\mathbf{c}' + m\mathbf{c}'_1$  holds for inelastic collisions, but the inelastic encounters are characterized by the relationship  $(\mathbf{g}' \cdot \mathbf{k}) = -\alpha(\mathbf{g} \cdot \mathbf{k})$  which relates the pre- and post-collisional relative velocities at collision. The parameter  $0 \leq \alpha \leq 1$  – here considered as a constant – refers to the normal restitution coefficient and  $\mathbf{k}$  is the unit vector directed

along the line which joins the molecules centers and pointing from center of the molecule labeled by the index 1 to the center of the molecule without label. The component of the velocity perpendicular to the collision vector  $\mathbf{k}$  does not change in inelastic collisions, so that  $\mathbf{k} \times \mathbf{g}' = \mathbf{k} \times \mathbf{g}$ . From the momentum conservation law it follows the relationships which give the post-collisional velocities in terms of the pre-collisional ones, as well as relationships which connect the relative velocities and their modulus:

$$\mathbf{c}' = \mathbf{c} + \frac{1+\alpha}{2}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \quad \mathbf{c}'_1 = \mathbf{c}_1 - \frac{1+\alpha}{2}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \quad (74)$$

$$\mathbf{g}' = \mathbf{g} - (1+\alpha)(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \quad g'^2 = g^2 - (1-\alpha^2)(\mathbf{g} \cdot \mathbf{k})^2. \quad (75)$$

In the case of elastic collisions,  $\alpha = 1$  and it follows the conservation of the kinetic energy  $g = g'$ .

A restitution collision with pre-collisional velocities  $(\mathbf{c}^*, \mathbf{c}_1^*)$  that corresponds to the post-collisional velocities  $(\mathbf{c}, \mathbf{c}_1)$  are characterized by the equations

$$\mathbf{c}^* = \mathbf{c} + \frac{1+\alpha}{2\alpha}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \quad \mathbf{c}_1^* = \mathbf{c}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \quad (76)$$

where  $\mathbf{k}^* = -\mathbf{k}$  and  $(\mathbf{g} \cdot \mathbf{k}) = -\alpha(\mathbf{g}^* \cdot \mathbf{k})$ .

For the determination of the Boltzmann equation we have to know the transformation of the volume elements  $d\mathbf{c}_1^* d\mathbf{c}^* = |J| d\mathbf{c}_1 d\mathbf{c}$ , where the modulus of the Jacobian is given by  $|J| = 1/\alpha$ . Hence, it follows the relationship

$$(\mathbf{g}^* \cdot \mathbf{k}^*) d\mathbf{c}^* d\mathbf{c}_1^* = \frac{1}{\alpha^2}(\mathbf{g} \cdot \mathbf{k}) d\mathbf{c} d\mathbf{c}_1, \quad (77)$$

and the Boltzmann equation for granular gases reads

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = \int \left( \frac{1}{\alpha^2} f_1^* f^* - f_1 f \right) d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1 \equiv \mathcal{Q}_I(f, f). \quad (78)$$

Above we have introduced the bilinear form

$$\mathcal{Q}_I(F, G) = \frac{1}{2} \int \left( \frac{1}{\alpha^2} F_1^* G^* + \frac{1}{\alpha^2} F^* G_1^* - F_1 G - F G_1 \right) d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1. \quad (79)$$

The balance equations for the mass density  $\varrho$ , for the momentum density  $\varrho v_i$  and for the specific internal energy  $\varepsilon = 3kT/2m$  are obtained from the Boltzmann equation (78) by multiplying it successively by  $m$ ,  $mc_i$  and  $mC^2/2$  and integrating the resulting equations. Hence, it follows

$$\mathcal{D}\varrho + \varrho \frac{\partial v_i}{\partial x_i} = 0, \quad \mathcal{D}v_i + \frac{\partial p_{ij}}{\partial x_j} = 0, \quad \mathcal{D}T + \frac{2}{3nk} \left( \frac{\partial q_i}{\partial x_i} + p_{ij} \frac{\partial v_i}{\partial x_j} \right) + T\zeta = 0. \quad (80)$$

We observe that the balance equation for the temperature (80)<sub>3</sub> has the additional term  $T\zeta$  which represents the energy loss due to inelastic collisions. The coefficient  $\zeta$ , known as cooling rate of the granular gas, is given by

$$\zeta = \frac{d^2 m(1-\alpha^2)}{12nkT} \int f_1 f(\mathbf{g} \cdot \mathbf{k})^3 d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}. \quad (81)$$



It is easy to verify from the above equation that the cooling rate vanishes for elastic collisions where  $\alpha = 1$ .

The methodology used to determine the distribution function by using the Chapman-Enskog method for granular gases is different from the one applied to a monatomic gas of elastic spherical molecules, since we have to take into account that there exists no equilibrium state characterized by a Maxwellian distribution function. The first difference, refers to the decomposition of the balance equations, since they are written as

$$\mathcal{D}_0 n = 0, \quad \mathcal{D}_1 n = -n \frac{\partial v_i}{\partial x_i}, \quad (82)$$

$$\mathcal{D}_0 v_i = 0, \quad \mathcal{D}_1 v_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}. \quad (83)$$

$$\mathcal{D}_0 T = -T \zeta^{(0)}, \quad \mathcal{D}_1 T = -T \zeta^{(1)} - \frac{2T}{3} \frac{\partial v_i}{\partial x_i}. \quad (84)$$

The first and the second approximation to the cooling rate in (84) read

$$\zeta^{(0)} = \frac{d^2 m (1 - \alpha^2)}{12 n k T} \int f_1^{(0)} f^{(0)} (\mathbf{g} \cdot \mathbf{k})^3 d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}, \quad (85)$$

$$\zeta^{(1)} = \frac{d^2 m (1 - \alpha^2)}{12 n k T} \int \left( f_1^{(1)} f^{(0)} + f_1^{(0)} f^{(1)} \right) (\mathbf{g} \cdot \mathbf{k})^3 d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}. \quad (86)$$

In terms of the parameter  $\Lambda$  we write the Boltzmann equation for granular gases (78) as

$$\frac{1}{\Lambda} \mathcal{D} f + C_i \frac{\partial f}{\partial x_i} = \frac{1}{\Lambda} \mathcal{Q}_I(f, f), \quad (87)$$

instead of the representation (24), indicating that the material time derivative and the collision term are of same order, while the spatial gradients are of higher order.

The two first integral equations for the determination of  $f^{(0)}$  and  $f^{(1)}$  are obtained by inserting the expansions (20) and (26) into the Boltzmann equation (87) and equating the equal powers of  $\Lambda$ , yielding

$$\mathcal{D}_0 f^{(0)} = \mathcal{Q}_I(f^{(0)}, f^{(0)}), \quad \mathcal{D}_1 f^{(0)} + \mathcal{D}_0 f^{(1)} + C_i \frac{\partial f^{(0)}}{\partial x_i} = 2 \mathcal{Q}_I(f^{(1)}, f^{(0)}). \quad (88)$$

In order to determine the first approximation of the distribution function the integral equation (88)<sub>1</sub> is written as

$$-T \zeta^{(0)} \frac{\partial f^{(0)}}{\partial T} = \mathcal{Q}_I(f^{(0)}, f^{(0)}), \quad (89)$$

through the elimination of the material time derivatives  $\mathcal{D}_0$  by using (82)<sub>1</sub>, (83)<sub>1</sub> and (84)<sub>1</sub>. We note that the solution of the integral equation (89) for the distribution function  $f^{(0)}$  is not a Maxwellian, but we may write  $f^{(0)}$  as a Maxwellian plus an expansion in Sonine polynomials of the peculiar velocity, i.e.,

$$f^{(0)} = f_M \left[ 1 + \sum_{n=1}^{\infty} a_n S_{\frac{1}{2}}^{(n)}(\beta C^2) \right], \quad \text{where} \quad f_M = n \left( \frac{\beta}{\pi} \right)^{\frac{3}{2}} e^{-\beta C^2}, \quad (90)$$

denotes the Maxwellian distribution function. The scalar coefficients  $a_n$  do not depend on the peculiar velocity  $\mathbf{C}$  and we suppose that this series expansion converges for large values of the peculiar velocity. We may approximate the representation of the distribution function (90) as

$$f^{(0)} = f_M \left[ 1 + a_1 S_{\frac{1}{2}}^{(1)}(\beta C^2) + a_2 S_{\frac{1}{2}}^{(2)}(\beta C^2) \right], \quad \text{where} \quad (91)$$

$$S_{\frac{1}{2}}^{(1)}(\beta C^2) = \frac{3}{2} - \beta C^2, \quad S_{\frac{1}{2}}^{(2)}(\beta C^2) = \frac{15}{8} - \frac{5}{2}\beta C^2 + \frac{1}{2}\beta^2 C^4. \quad (92)$$

If we insert the representation (91) into the definition of the temperature (9) and integrate the resulting equation we get that  $a_1 = 0$ .

For the determination of the coefficient  $a_2$  we proceed as follows. First the product of the distribution functions is written as

$$f^{(0)} f_1^{(0)} = n^2 \left( \frac{\beta}{\pi} \right)^3 e^{-\beta \left( 2G^2 + \frac{g^2}{2} \right)} \left\{ 1 + a_2 \left[ \frac{15}{4} - \frac{5\beta}{2} \left( 2G^2 + \frac{g^2}{2} \right) + \frac{\beta^2}{2} \left( 2G^4 + G^2 g^2 + \frac{g^4}{8} + 2(\mathbf{G} \cdot \mathbf{g})^2 \right) \right] \right\}, \quad (93)$$

by neglecting all product of the coefficient  $a_2$  and changing the velocity variables  $(\mathbf{C}, \mathbf{C}_1)$  by the relative and the center of mass velocities  $(\mathbf{g}, \mathbf{G})$ . Next, we multiply the integral equation (89) by an arbitrary function of the peculiar velocity  $\psi(C^2)$  and integrate the resulting equation over all values of  $\mathbf{c}$ , yielding

$$- \int \psi(C^2) T \zeta^{(0)} \frac{\partial f^{(0)}}{\partial T} d\mathbf{c} = \frac{1}{2} \int [\psi(C_1'^2) + \psi(C'^2) - \psi(C_1^2) - \psi(C^2)] f_1^{(0)} f^{(0)} d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}. \quad (94)$$

If we choose  $\psi(C^2) = 1$  and  $\psi(C^2) = C^2$  in (94) the integration of the resulting equations lead to identities. However, if we choose  $\psi(C^2) = C^4$  in (94) and use the relationship

$$C_1'^4 + C'^4 - C_1^4 - C^4 = 2(\alpha + 1)^2 (\mathbf{g} \cdot \mathbf{k})^2 (\mathbf{G} \cdot \mathbf{k})^2 + \frac{(\alpha^2 - 1)^2}{8} (\mathbf{g} \cdot \mathbf{k})^4 + (\alpha^2 - 1) (\mathbf{g} \cdot \mathbf{k})^2 G^2 + \frac{(\alpha^2 - 1)}{4} (\mathbf{g} \cdot \mathbf{k})^2 g^2 - 4(\alpha + 1) (\mathbf{g} \cdot \mathbf{k}) (\mathbf{G} \cdot \mathbf{k}) (\mathbf{G} \cdot \mathbf{g}), \quad (95)$$

we obtain through the integration of the resulting equation that the coefficient  $a_2$  has the form

$$a_2 = \frac{16(1 - \alpha)(1 - 2\alpha^2)}{81 - 17\alpha + 30\alpha^2(1 - \alpha)}. \quad (96)$$

We may infer from the above equation that the coefficient  $a_2$  vanishes for elastic collisions where  $\alpha = 1$ .

From the knowledge of the coefficient  $a_2$ , the distribution function (91) is written as

$$f^{(0)} = f_M \left[ 1 + \frac{16(1 - \alpha)(1 - 2\alpha^2)}{81 - 17\alpha + 30\alpha^2(1 - \alpha)} S_{\frac{1}{2}}^{(2)}(\beta C^2) \right]. \quad (97)$$

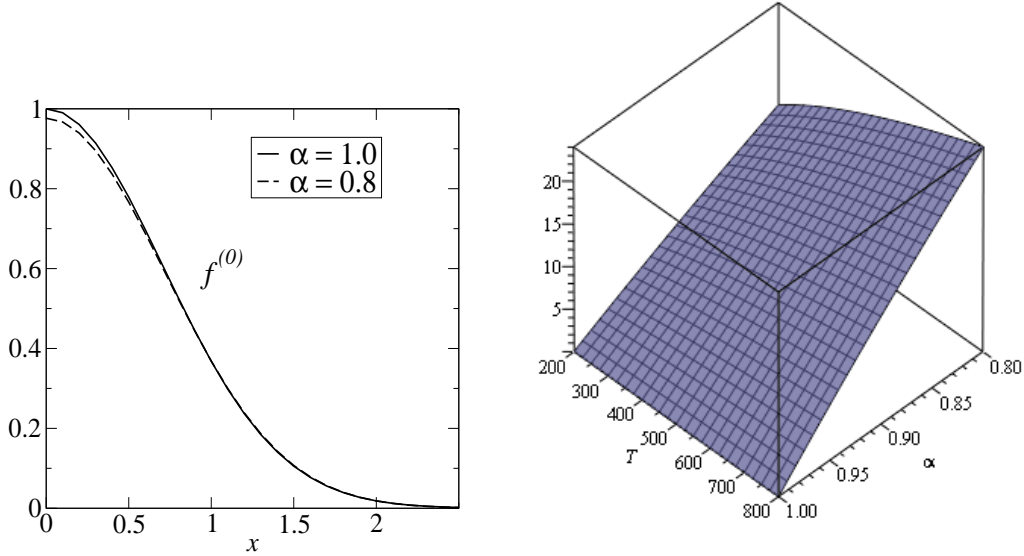


Figure 2: Left frame: dimensionless distribution function  $\pi^{3/2}f^{(0)}/n\beta^{3/2}$  as a function of  $x = \beta C^2$  for  $\alpha = 1$  and  $0.8$ . Right frame: cooling rate  $\zeta^{(0)}/nd^2$  versus the temperature and the normal restitution coefficient.

If we substitute the distribution function (97) into the expression for the cooling rate (85) and integrate the resulting equation we get that

$$\zeta^{(0)} = \frac{4}{3}nd^2(1 - \alpha^2)\sqrt{\frac{\pi kT}{m}} \left[ 1 + \frac{3(1 - \alpha)(1 - 2\alpha^2)}{81 - 17\alpha + 30\alpha^2(1 - \alpha)} \right]. \quad (98)$$

In the left frame of Figure 2 it is represented the dimensionless distribution function  $\pi^{3/2}f^{(0)}/n\beta^{3/2}$  as a function of  $x = \beta C^2$  and  $\alpha$ . We may infer from this figure that the peak of the curve decreases by decreasing the normal restitution coefficient. In the right frame of the Figure 2 it is plotted the cooling rate  $\zeta^{(0)}/nd^2$  versus the temperature and the normal restitution coefficient. We note that the cooling rate increases by decreasing the normal restitution coefficient and by increasing the temperature. Furthermore, the cooling rate is zero when  $\alpha = 1$ , i.e., for elastic collisions of the molecules.

The determination of the second approximation to the distribution function  $f^{(1)}$  from the integral equation (88)<sub>2</sub> is more involved and the details of such kind of calculation will not be given here. From this integral equation we may conclude that  $f^{(1)}$  is a function of the thermodynamic forces: deviator of the velocity gradient  $\partial v_{\langle i}/\partial x_{j\rangle}$ , gradient of the particle number density  $\partial n/\partial x_i$  and gradient of temperature  $\partial T/\partial x_i$ . Hence, we may write the second approximation  $f^{(1)}$  as

$$f^{(1)} = f_M \left[ \gamma_1 S_{\frac{3}{2}}^{(1)}(C^2) C_i \frac{\partial \ln T}{\partial x_i} + \gamma_2 S_{\frac{5}{2}}^{(0)}(C^2) C_i C_j \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \gamma_3 S_{\frac{3}{2}}^{(1)}(C^2) C_i \frac{\partial \ln n}{\partial x_i} \right], \quad (99)$$

where  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$  are scalar coefficients that depend on  $n$ ,  $T$  and  $\alpha$ . If we solve the integral equation (88)<sub>2</sub> we get that the coefficients of the distribution function  $f^{(1)}$  are

given by

$$\gamma_1 = \frac{15}{d^2 n (9 + 7\alpha)(\alpha + 1)} \sqrt{\frac{m}{\pi k T}}, \quad \gamma_2 = \frac{-15}{2d^2 n (13 - \alpha)(\alpha + 1)} \sqrt{\frac{1}{\pi} \left(\frac{m}{k T}\right)^3}, \quad (100)$$

$$\gamma_3 = \frac{300(1 - \alpha)}{d^2 n (9 + 7\alpha)(\alpha + 1)(19 - 3\alpha)} \sqrt{\frac{m}{\pi k T}}. \quad (101)$$

We obtain the constitutive equations for the pressure tensor and heat flux vector through the substitution of the distribution function (99) into their definitions and by integrating the resulting equations. Hence, it follows the laws of Navier-Stokes and Fourier

$$p_{ij} = \int m C_i C_j f d\mathbf{c} = p \delta_{ij} - 2\mu \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}, \quad (102)$$

$$q_i = \int \frac{m}{2} C^2 C_i f d\mathbf{c} = -\lambda \frac{\partial T}{\partial x_i} - \vartheta T \frac{\partial \ln n}{\partial x_i}. \quad (103)$$

Above, the coefficients of shear viscosity  $\mu$ , thermal conductivity  $\lambda$  and the one associated with the gradient of particle number density  $\vartheta$  read

$$\mu = \frac{15}{2d^2 (13 - \alpha)(\alpha + 1)} \sqrt{\frac{m k T}{\pi}}, \quad \lambda = \frac{75}{2d^2 (9 + 7\alpha)(\alpha + 1)} \frac{k}{m} \sqrt{\frac{m k T}{\pi}}, \quad (104)$$

$$\vartheta = \frac{750(1 - \alpha)}{d^2 n (9 + 7\alpha)(\alpha + 1)(19 - 3\alpha)} \frac{k}{m} \sqrt{\frac{m k T}{\pi}}. \quad (105)$$

We note that the coefficients are proportional to  $\sqrt{T}$  so that they increase by increasing of the temperature. With respect to the normal restitution coefficient  $\alpha$ , we may infer that the coefficients also increase by decreasing the values of  $\alpha$ . Furthermore, by considering elastic collisions where  $\alpha = 1$  the coefficients of shear viscosity and thermal conductivity reduce to the first approximations (73)<sub>1,2</sub>, while the coefficient  $\vartheta$  vanishes.

## 5 Grad's Method of Moments

Let us consider a thirteen scalar field description of a rarefied gas within Grad's moment method. In this case the fields are the mass density  $\varrho(\mathbf{x}, t)$ , the hydrodynamic velocity  $v_i(\mathbf{x}, t)$ , the pressure tensor  $p_{ij}(\mathbf{x}, t)$  and the heat flux vector  $q_i(\mathbf{x}, t)$  and their balance equations read

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \quad (106)$$

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial (\varrho v_i v_j + p_{ij})}{\partial x_j} = 0, \quad (107)$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial (p_{ijk} + p_{ij} v_k)}{\partial x_k} + p_{ki} \frac{\partial v_j}{\partial x_k} + p_{kj} \frac{\partial v_i}{\partial x_k} = P_{ij}, \quad (108)$$

$$\frac{\partial q_i}{\partial t} + \frac{\partial (q_{ij} + q_i v_j)}{\partial x_j} + p_{ijk} \frac{\partial v_j}{\partial x_k} + q_j \frac{\partial v_i}{\partial x_j} - \frac{p_{ki}}{\varrho} \frac{\partial p_{kj}}{\partial x_j} - \frac{1}{2} \frac{p_{rr}}{\varrho} \frac{\partial p_{ij}}{\partial x_j} = Q_i. \quad (109)$$

We note that the two first equations above represent the balance equations of mass density (11) and moment density (12), while the balance equation for the pressure tensor (108) was obtained from (14) by taking  $N = 2$ . The balance equation for the heat flux vector (109) follows from (14) by taking  $N = 3$ , contracting two indices and dividing the resulting equation by 2. Moreover, we have introduced the following definitions in (108) and (109):

$$p_{ijk} = \int m C_i C_j C_k f d\mathbf{c}, \quad P_{ij} = \int m (C'_i C'_j - C_i C_j) f f_1 d\Gamma, \quad (110)$$

$$q_{ij} = \int \frac{m}{2} C^2 C_i C_j f d\mathbf{c}, \quad Q_i = \int \frac{m}{2} (C'^2 C'_i - C^2 C_i) f f_1 d\Gamma. \quad (111)$$

The tensors  $p_{ijk}$  and  $q_{ij}$  represent non-convective fluxes for the pressure tensor and heat flux vector, respectively, while  $P_{ij}$  and  $Q_i$  refer to their production terms. In the above equations we have introduced the abbreviation  $d\Gamma = g b db d\varepsilon d\mathbf{c}_1 d\mathbf{c}$ .

If we analyze the system of partial differential equations (106) – (109) we may infer that it cannot be considered a system of field equations for the basic fields  $\varrho$ ,  $v_i$ ,  $p_{ij}$  and  $q_i$ . In fact, to obtain a closed system of differential equations, the non-convective fluxes  $p_{ijk}$ ,  $q_{ij}$  and the production terms  $P_{ij}$ ,  $Q_i$  must be expressed in terms of the basic fields, and for this end we must express the distribution function in terms of the thirteen scalar fields  $\varrho$ ,  $v_i$ ,  $p_{ij}$  and  $q_i$ .

In Grad's moment method the distribution function is expanded in series of tensorial Hermite polynomials  $H_{i_1 i_2 \dots i_N}$  ( $N = 0, 1, 2, \dots$ ) as follows

$$f = f^{(0)} \left( aH + a_i H_i + \frac{1}{2!} a_{ij} H_{ij} + \dots + \frac{1}{N!} a_{i_1 i_2 \dots i_N} H_{i_1 i_2 \dots i_N} + \dots \right), \quad (112)$$

where  $a_{i_1 i_2 \dots i_N}$  ( $N = 0, 1, 2, \dots$ ) are tensorial coefficients that depend on  $\mathbf{x}$  and  $t$ . Furthermore, the Maxwellian distribution function  $f^{(0)}$  is written as

$$f^{(0)} = n \left( \frac{\beta}{\pi} \right)^{\frac{3}{2}} e^{-\beta C^2} = n \left( \frac{m}{kT} \right)^{\frac{3}{2}} \omega(\boldsymbol{\xi}), \quad (113)$$

where  $\omega(\boldsymbol{\xi})$  is the weight function:

$$\omega(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\xi^2/2}, \quad \text{with} \quad \xi_i = \sqrt{\frac{m}{kT}} C_i. \quad (114)$$

On the basis of the weight function, the tensorial Hermite polynomials are written as

$$H_{i_1 i_2 \dots i_N}(\boldsymbol{\xi}) = \frac{(-1)^N}{\omega} \frac{\partial^N \omega}{\partial \xi_{i_1} \partial \xi_{i_2} \dots \partial \xi_{i_N}}, \quad (115)$$

being orthogonal with respect to  $\omega(\boldsymbol{\xi})$ , i.e.,

$$\int \omega(\boldsymbol{\xi}) H_{i_1 i_2 \dots i_N}(\boldsymbol{\xi}) H_{j_1 j_2 \dots j_M}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \delta_{MN} \Delta_{i_1 j_1 i_2 j_2 \dots i_N j_N}. \quad (116)$$

In the above equation,  $\Delta_{i_1 j_1 i_2 j_2 \dots i_N j_N}$  represents the sum  $\Delta_{i_1 j_1 i_2 j_2 \dots i_N j_N} = \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_N j_N} +$  (all permutations of the  $j$ 's indices) and  $\delta_{MN}$  is Kronecker's symbol.

From (115) we can obtain the first four tensorial Hermite polynomials, namely,

$$H(\boldsymbol{\xi}) = 1, \quad H_i(\boldsymbol{\xi}) = \xi_i, \quad H_{ij}(\boldsymbol{\xi}) = \xi_i \xi_j - \delta_{ij}, \quad (117)$$

$$H_{ijk}(\boldsymbol{\xi}) = \xi_i \xi_j \xi_k - (\xi_i \delta_{jk} + \xi_j \delta_{ik} + \xi_k \delta_{ij}). \quad (118)$$

In terms of the tensorial Hermite polynomials we can write the thirteen moments of the distribution function as

$$\varrho = \int m \left( \frac{kT}{m} \right)^{\frac{3}{2}} f H d\boldsymbol{\xi}, \quad 0 = \int m \left( \frac{kT}{m} \right)^2 f H_i d\boldsymbol{\xi}, \quad (119)$$

$$T = \frac{m}{3k\varrho} \int m \left( \frac{kT}{m} \right)^{\frac{5}{2}} f (H_{ii} + 3H) d\boldsymbol{\xi}, \quad (120)$$

$$p_{\langle ij \rangle} = \int m \left( \frac{kT}{m} \right)^{\frac{5}{2}} f \left( H_{ij} - \frac{1}{3} H_{rr} \delta_{ij} \right) d\boldsymbol{\xi}, \quad q_i = \int \frac{m}{2} \left( \frac{kT}{m} \right)^3 f (H_{ijj} + 5H_i) d\boldsymbol{\xi}. \quad (121)$$

We recall that the temperature is given by  $T = 2mp_{rr}/3k\varrho$ .

The coefficients  $a_{i_1 i_2 \dots i_N}$  that appear in the distribution function (112) are determined from the definition of the moments of the distribution function. Furthermore, for a thirteen moment theory the distribution function (112) is written as

$$f = f^{(0)} \left( aH + a_i H_i + \frac{1}{2} a_{ij} H_{ij} + \frac{1}{10} a_{rri} H_{ssi} \right), \quad (122)$$

where  $a$ ,  $a_i$ ,  $a_{ij}$  and  $a_{rri}$  represent thirteen scalar coefficients to be determined. We note that the term  $\frac{1}{3!} a_{ijk} H_{ijk}$  was decomposed according to

$$\frac{1}{6} \left[ a_{\langle ijk \rangle} + \frac{1}{5} (a_{rri} \delta_{jk} + a_{rrj} \delta_{ik} + a_{rrk} \delta_{ij}) \right] H_{ijk} = \frac{1}{6} a_{\langle ijk \rangle} H_{ijk} + \frac{1}{10} a_{rri} H_{ssi}, \quad (123)$$

and the part associated with the third order tensor  $\frac{1}{6} a_{\langle ijk \rangle} H_{ijk}$  was not taken into account.

If we insert now the distribution function (122) into the definitions of the moments (119) – (121), integrate the resulting equations and use the orthogonality properties of the tensorial Hermite polynomials (116), we obtain that

$$a = 1, \quad a_i = 0, \quad a_{rr} = 0, \quad a_{\langle ij \rangle} = \frac{p_{\langle ij \rangle}}{p}, \quad a_{rri} = \frac{2q_i}{p} \sqrt{\frac{m}{kT}}. \quad (124)$$

Hence, the distribution function for the thirteen moments (with  $\beta = m/2kT$ ) becomes

$$f = f^{(0)} \left\{ 1 + \frac{2\beta^2}{\varrho} \left[ p_{\langle ij \rangle} C_i C_j + \frac{4}{5} q_i C_i \left( \beta C^2 - \frac{5}{2} \right) \right] \right\}, \quad (125)$$

which is the so-called Grad's distribution function.

In order to determine the non-convective fluxes  $p_{ijk}$  and  $q_{ij}$  we insert Grad's distribution function (125) into the definitions (110)<sub>1</sub> and (111)<sub>1</sub> and obtain by integrating the resulting equations:

$$p_{ijk} = \frac{2}{5} (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}), \quad q_{ij} = \frac{5p^2}{2\varrho} \delta_{ij} + \frac{7p}{2\varrho} p_{\langle ij \rangle}. \quad (126)$$

For the calculation of the production terms  $P_{ij}$  and  $Q_i$  we change the integration variables  $(\mathbf{c}, \mathbf{c}_1) \mapsto (\mathbf{g}, \mathbf{G})$  so that in these new variables  $(110)_2$  and  $(111)_2$  become

$$P_{ij} = \int \frac{m}{2} \left[ \frac{1}{2} (g'_i g'_j - g_i g_j) - G_i (g'_j - g_j) - G_j (g'_i - g_i) \right] f f_1 d\Gamma, \quad (127)$$

$$Q_i = \int \frac{m}{2} \left[ \frac{1}{2} G_r (g'_r g'_i - g_r g_i) - \frac{1}{2} \left( G^2 + \frac{1}{4} g^2 \right) (g'_i - g_i) - G_i G_r (g'_r - g_r) \right] f f_1 d\Gamma, \quad (128)$$

where the product of Grad's distribution functions in a linearized theory reads

$$f f_1 = n^2 \left( \frac{\beta}{\pi} \right)^3 e^{-(2\beta G^2 + \frac{\beta}{2} g^2)} \left\{ 1 + \frac{4\beta^2}{\varrho} p_{\langle kl \rangle} \left[ G_k G_l + \frac{1}{4} g_k g_l \right] \right. \quad (129)$$

$$\left. + \frac{8\beta^2}{5\varrho} q_j \left[ 2G_j \left( \beta G^2 + \frac{\beta}{4} g^2 - \frac{5}{2} \right) + \beta g_j G_k g_k \right] \right\}. \quad (130)$$

The integration of (127) and (128), yields

$$P_{ij} = -\frac{p_{\langle ij \rangle}}{\tau_r}, \quad Q_i = -\frac{2q_i}{3\tau_r}, \quad \text{where} \quad \tau_r = \frac{5}{16n} \sqrt{\frac{m}{\pi kT}} \frac{1}{\Omega^{(2,2)}}, \quad (131)$$

represents the relaxation time of the pressure deviator.

Once we know the constitutive equations for the non-convective fluxes (126) and production terms (131) as functions of the thirteen basic fields  $\varrho$ ,  $v_i$ ,  $p_{ij}$  and  $q_i$ , a system of linearized thirteen scalar field equations can be obtained from (106) – (109), namely

$$\mathcal{D}\varrho + \varrho \frac{\partial v_i}{\partial x_i} = 0, \quad (132)$$

$$\varrho \mathcal{D}v_i + \frac{\partial p}{\partial x_i} + \frac{\partial p_{\langle ij \rangle}}{\partial x_j} = 0, \quad (133)$$

$$\frac{3}{2} n k \mathcal{D}T + \frac{\partial q_i}{\partial x_i} + p \frac{\partial v_i}{\partial x_i} + p_{\langle ij \rangle} \frac{\partial v_i}{\partial x_j} = 0, \quad (134)$$

$$\mathcal{D}p_{\langle ij \rangle} + \frac{4}{5} \frac{\partial q_{\langle i}}{\partial x_{j \rangle}} + \underline{2p \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}} = -\frac{p_{\langle ij \rangle}}{\tau_r}, \quad (135)$$

$$\mathcal{D}q_i + \frac{p}{\varrho} \frac{\partial p_{\langle ik \rangle}}{\partial x_k} + \underline{\frac{5}{2} \frac{k}{m} p \frac{\partial T}{\partial x_i}} = -\frac{2q_i}{3\tau_r}. \quad (136)$$

If we restrict ourselves to a five field theory described by the fields of mass density, momentum density and temperature, the corresponding balance equations are given by (132), (133) and (134). In this case the pressure deviator and the heat flux vector are no longer variables, just constitutive quantities. We may use the remaining eight equations (135) and (136) in order to obtain the constitutive equations for the pressure deviator and for the heat flux vector, by considering only the equilibrium values of the pressure deviator  $p_{\langle ij \rangle} = 0$  and of the heat flux vector  $q_i = 0$  on the left-hand sides of (135) and (136). Hence, only the underlined terms remain, and we obtain

$$p_{\langle ij \rangle} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}, \quad \text{where} \quad \mu = p\tau_r = \frac{5}{16} \sqrt{\frac{mkT}{\pi}} \frac{1}{\Omega^{(2,2)}}, \quad (137)$$

$$q_i = -\lambda \frac{\partial T}{\partial x_i}, \quad \text{where} \quad \lambda = \frac{15k}{4m} p\tau_r = \frac{75k}{64m} \sqrt{\frac{mkT}{\pi}} \frac{1}{\Omega^{(2,2)}}. \quad (138)$$

Equations (137) and (138) represent the laws of Navier-Stokes and Fourier, respectively, and the coefficients of shear viscosity  $\mu$  and thermal conductivity  $\lambda$  correspond to the first approximation to these coefficients found by applying the Chapman-Enskog method (see (68)).

## 6 Chapman-Enskog-Grad Combined Method

We describe now another method to obtain the constitutive equations for the pressure deviator and for the heat flux vector which combines the features of Chapman-Enskog and Grad's methods. In this method neither a solution of the integral equation is needed – as in the Chapman-Enskog method – nor the field equations for the moments of the distribution function are used – as in Grad's method.

To begin with we observe that in the representation  $f = f^{(0)}(1 + \phi)$  the deviation of the Maxwellian distribution function  $\phi$  in the Chapman-Enskog method – which is a unknown quantity – can be written in terms of known quantities. These known quantities can be chosen as the thirteen scalar fields of mass density, hydrodynamic velocity, temperature, pressure deviator and heat flux vector which appear in Grad's distribution function (125), namely,

$$\phi = \frac{2\beta^2}{\varrho} \left[ p_{\langle ij \rangle} C_i C_j + \frac{4}{5} q_i C_i \left( \beta C^2 - \frac{5}{2} \right) \right]. \quad (139)$$

In the framework of the Chapman-Enskog-Grad combined method we insert the representation (139) into the non-homogeneous integral equation of the Chapman-Enskog method (34), so that it becomes an equation for the determination of the pressure deviator  $p_{\langle ij \rangle}$  and of the heat flux vector  $q_i$ . This equation reads

$$f^{(0)} \left\{ \frac{1}{T} \left( \beta C^2 - \frac{5}{2} \right) C_k \frac{\partial T}{\partial x_k} + 2\beta C_k C_l \frac{\partial v_{\langle k}}{\partial x_{l \rangle}} \right\} = \frac{2\beta^2}{\varrho} p_{\langle kl \rangle} \mathcal{I}[C_k C_l] + \frac{8\beta^2}{5\varrho} q_k \mathcal{I} \left[ \left( \beta C^2 - \frac{5}{2} \right) C_k \right]. \quad (140)$$

For the determination of the pressure deviator, we multiply (140) by  $C_{\langle i} C_{j \rangle}$  and integrate the resulting equation over all values of  $\mathbf{c}$ , yielding

$$2 \frac{p}{m} \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = \frac{2\beta^2}{\varrho} p_{\langle kl \rangle} \int C_{\langle i} C_{j \rangle} \mathcal{I}[C_k C_l] d\mathbf{c}. \quad (141)$$

To solve the above equation for the pressure deviator  $p_{\langle kl \rangle}$ , we introduce the integral

$$I_{ijkl} = \int C_i C_j \mathcal{I}[C_k C_l] d\mathbf{c} = A_1 \delta_{ij} \delta_{kl} + A_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (142)$$

which was written in terms of Kronecker's symbol with coefficients  $A_1$  and  $A_2$ . If we contract the integral  $I_{ijkl}$  in two different manners, namely,  $I_{rrss} = 9A_1 + 6A_2$  and  $I_{rsrs} = 3A_1 + 12A_2$  we get that

$$I_{\langle ij \rangle kl} = \frac{3I_{rsrs} - I_{rrss}}{15} \delta_{k \langle i} \delta_{j \rangle l}. \quad (143)$$



We use now the relationship (143) to write (141) as

$$2 \frac{p}{m} \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = \frac{2\beta^2}{5\varrho} p_{\langle ij \rangle} \int C_{\langle r} C_s \mathcal{I}[C_{\langle r} C_s \rangle] d\mathbf{c} = -\frac{2\beta^2}{5\varrho} \beta^{(0,0)} p_{\langle ij \rangle}, \quad (144)$$

where  $\beta^{(0,0)}$  is given by (65). Hence, it follows from (144) Navier-Stokes' law

$$p_{\langle ij \rangle} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}, \quad \text{where} \quad \mu = \frac{5}{16} \sqrt{\frac{mkT}{\pi}} \frac{1}{\Omega^{(2,2)}}, \quad (145)$$

is the coefficient of shear viscosity.

We determine the heat flux vector by following the same methodology. We multiply (140) by  $C^2 C_i$  and the integrate of the resulting equation over all values of  $\mathbf{c}$ , resulting

$$5 \frac{kp}{m^2} \frac{\partial T}{\partial x_i} = \frac{8\beta^2}{5\varrho} q_k \int C^2 C_i \mathcal{I} \left[ \left( \beta C^2 - \frac{5}{2} \right) C_k \right] d\mathbf{c} = -\frac{8\beta^2}{15\varrho} \alpha^{(1,1)} q_i, \quad (146)$$

thanks to the following relationship

$$I_{ik} = \int C^2 C_i \mathcal{I} \left[ \left( \beta C^2 - \frac{5}{2} \right) C_k \right] d\mathbf{c} = \frac{I_{rr}}{3} \delta_{ik}. \quad (147)$$

Hence, Fourier's law follows from (146)

$$q_i = -\lambda \frac{\partial T}{\partial x_i}, \quad \text{where} \quad \lambda = \frac{75}{64} \frac{k}{m} \sqrt{\frac{mkT}{\pi}} \frac{1}{\Omega^{(2,2)}}, \quad (148)$$

is the coefficient of thermal conductivity. Above, we have used the relationship  $\alpha^{(1,1)} = \beta^{(0,0)}$  which was obtained in the Chapman-Enskog method.

We can extended this method to obtain the successive approximations to the transport coefficients, by introducing traceless second order tensors  $p_{\langle ij \rangle}^{(s)}$  and vectors  $q_i^{(s)}$  defined by

$$p_{\langle ij \rangle}^{(s)} = 15 \frac{(2s)!!}{(2s+5)!!} \int m S_{\frac{5}{2}}^{(s)} C_{\langle i} C_{j \rangle} f d\mathbf{c}, \quad q_i^{(s)} = -\frac{15(2s)!!}{4\beta(2s+3)!!} \int m S_{\frac{3}{2}}^{(s)} C_i f d\mathbf{c}, \quad (149)$$

where  $p_{\langle ij \rangle}^{(0)} = p_{\langle ij \rangle}$  represents the pressure deviator and  $q_i^{(0)} = q_i$  the heat flux vector, while the the higher-order moments  $p_{\langle ij \rangle}^{(s)} (s > 0)$  and  $q_i^{(s)} (s > 1)$  do not have standard designations.

For the moments of the distribution function characterized by  $\varrho$ ,  $v_i$ ,  $T$ ,  $p_{\langle ij \rangle}^{(s)}$  and  $q_i^{(s)}$  the distribution function reads

$$f = f^{(0)} \left\{ 1 + \frac{2\beta^2}{\varrho} \sum_{s=0}^{\infty} S_{\frac{5}{2}}^{(s)} C_{\langle i} C_{j \rangle} p_{\langle ij \rangle}^{(s)} - \frac{8\beta^2}{5\varrho} \sum_{s=0}^{\infty} S_{\frac{3}{2}}^{(s)} C_i q_i^{(s)} \right\}, \quad (150)$$

so that (140) becomes

$$f^{(0)} \left\{ -\frac{1}{T} S_{\frac{3}{2}}^{(1)} C_k \frac{\partial T}{\partial x_k} + 2\beta S_{\frac{5}{2}}^{(0)} C_k C_l \frac{\partial v_{\langle k}}{\partial x_{l \rangle}} \right\} = \frac{2\beta^2}{\varrho} \sum_{s=0}^{\infty} p_{\langle kl \rangle}^{(s)} \mathcal{I} \left[ S_{\frac{5}{2}}^{(s)} C_k C_l \right] - \frac{8\beta^2}{5\varrho} \sum_{s=0}^{\infty} q_k^{(s)} \mathcal{I} \left[ S_{\frac{3}{2}}^{(s)} C_k \right]. \quad (151)$$

The multiplication of (151) by  $\beta S_{\frac{5}{2}}^{(s)} C_{\langle i} C_{j \rangle}$  and integration of the resulting equation leads to

$$-\frac{5}{2} \frac{\varrho^2}{m\beta^3} \delta^{(0,r)} \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = \sum_{s=0}^{\infty} \beta^{(r,s)} p_{\langle ij \rangle}^{(s)}. \quad (152)$$

If we solve the above equation for the pressure deviator  $p_{\langle ij \rangle}$  it follows the Navier-Stokes law

$$p_{\langle ij \rangle} = -2 [\mu]_p \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}, \quad \text{where} \quad [\mu]_p = \frac{5}{4} \frac{\varrho^2}{m\beta^3} (\beta^{-1})^{(0,0)} \quad (153)$$

is the coefficient of shear viscosity in the  $p^{\text{th}}$  successive approximation. In  $(153)_2 (\beta^{-1})^{(0,0)}$  represents the first row and first column of the inverse of the  $p \times p$  matrix  $\beta^{(r,s)}$ .

If we multiply (151) by  $\beta S_{\frac{3}{2}}^{(s)} C_i$  and integrate the resulting equation we get

$$-\frac{75}{16} \frac{n^2 k}{\beta} \delta^{(1,r)} \frac{\partial T}{\partial x_i} = \sum_{s=0}^{\infty} \alpha^{(r,s)} q_i^{(s)}, \quad (154)$$

which can be solved for the heat flux vector  $q_i$ , yielding Fourier's law:

$$q_i = -[\lambda]_p \frac{\partial T}{\partial x_i}, \quad \text{where} \quad [\lambda]_p = \frac{75}{16} \frac{n^2 k}{\beta} (\alpha^{-1})^{(1,1)}. \quad (155)$$

The coefficient of thermal conductivity  $[\lambda]_p$  in the  $p^{\text{th}}$  successive approximation is given in terms of the element  $(\alpha^{-1})^{(1,1)}$  of the inverse matrix.

The successive approximations of the transport coefficients of shear viscosity  $(153)_2$  and thermal conductivity  $(155)_2$  are the same as those which follow from the Chapman-Enskog method.

## 7 Fourteen Moment Theory for Granular Gases

We may characterize a macroscopic state of the granular gas by the fourteen fields of mass density  $\varrho$ , hydrodynamic velocity  $v_i$ , pressure tensor  $p_{ij}$ , heat flux vector  $q_i$  and contracted fourth moment  $p_{iijj}$  which are defined by

$$\varrho = \int m f d\mathbf{c}, \quad \varrho v_i = \int m c_i f d\mathbf{c}, \quad p_{ij} = \int m C_i C_j f d\mathbf{c}, \quad (156)$$

$$q_i = \int \frac{m}{2} C^2 C_i f d\mathbf{c}, \quad p_{iijj} = \int m C^4 f d\mathbf{c}. \quad (157)$$

The balance equations for the fourteen fields are obtained from the Boltzmann equation (78) by multiplying it successively by  $m$ ,  $m c_i$ ,  $m C_i C_j$ ,  $m C^2 C_i / 2$  and  $m C^4$  and integrating

the resulting equations. Hence, it follows

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \quad (158)$$

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\varrho v_i v_j + p_{ij}) = 0, \quad (159)$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_k} (p_{ijk} + p_{ij} v_k) + p_{ki} \frac{\partial v_j}{\partial x_k} + p_{kj} \frac{\partial v_i}{\partial x_k} = P_{ij}, \quad (160)$$

$$\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j} (q_{ij} + q_i v_j) + p_{ijk} \frac{\partial v_j}{\partial x_k} + q_j \frac{\partial v_i}{\partial x_j} - \frac{p_{ki}}{\varrho} \frac{\partial p_{kj}}{\partial x_j} - \frac{1}{2} \frac{p_{rr}}{\varrho} \frac{\partial p_{ij}}{\partial x_j} = Q_i, \quad (161)$$

$$\frac{\partial p_{iijj}}{\partial t} + \frac{\partial}{\partial x_j} (p_{iijjk} + p_{iijj} v_j) + 8q_{ik} \frac{\partial v_i}{\partial x_k} - \frac{8}{\varrho} q_i \frac{\partial p_{ik}}{\partial x_k} = P. \quad (162)$$

In the above equations the new moments of the distribution function are defined by

$$p_{ijk} = \int m C_i C_j C_k f d\mathbf{c}, \quad q_{ij} = \int \frac{m}{2} C^2 C_i C_j f d\mathbf{c}, \quad p_{iijjk} = \int m C^4 C_k f d\mathbf{c}, \quad (163)$$

where  $q_{ii} = p_{iijj}/2$ . Furthermore, the production terms read

$$P_{ij} = \frac{1}{2} \int m \left( C_i^{1'} C_j^{1'} + C_i' C_j' - C_i^1 C_j^1 - C_i C_j \right) f_1 f d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}, \quad (164)$$

$$Q_i = \frac{1}{2} \int \frac{m}{2} \left( C_1'^2 C_i^{1'} + C_1'^2 C_i' - C_1^2 C_i^1 - C^2 C_i \right) f_1 f d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}, \quad (165)$$

$$P = \frac{1}{2} \int m \left( C_1'^4 + C'^4 - C_1^4 - C^4 \right) f_1 f d^2(\mathbf{g} \cdot \mathbf{k}) d\mathbf{k} d\mathbf{c}_1 d\mathbf{c}. \quad (166)$$

We can decompose the balance equation for the pressure tensor (160) in its trace and traceless parts by introducing the traceless tensors  $p_{\langle ij \rangle}$  and  $p_{\langle ijk \rangle}$  defined by

$$p_{ij} = p_{\langle ij \rangle} + p \delta_{ij}, \quad p_{ijk} = p_{\langle ijk \rangle} + \frac{2}{5} (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}), \quad (167)$$

where  $p = nkT$  is the hydrostatic pressure. Hence, we obtain from (160) the equations:

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} + \frac{2}{3nk} \left( \frac{\partial q_i}{\partial x_i} + p \frac{\partial v_i}{\partial x_i} + p_{\langle ij \rangle} \frac{\partial v_i}{\partial x_j} \right) + T \zeta = 0, \quad (168)$$

$$\begin{aligned} & \frac{\partial p_{\langle ij \rangle}}{\partial t} + \frac{\partial}{\partial x_k} (p_{\langle ijk \rangle} + p_{\langle ij \rangle} v_k) + p_{\langle ki \rangle} \frac{\partial v_j}{\partial x_k} + p_{\langle kj \rangle} \frac{\partial v_i}{\partial x_k} - \frac{2}{3} p_{\langle kr \rangle} \frac{\partial v_r}{\partial x_k} \delta_{ij} \\ & + \frac{4}{5} \frac{\partial q_{\langle i \rangle}}{\partial x_j} + 2p \frac{\partial v_{\langle i \rangle}}{\partial x_j} = P_{\langle ij \rangle}. \end{aligned} \quad (169)$$

Equation (168) is the balance equation for the temperature with  $\zeta$  denoting the cooling rate (81), while (169) is the balance equation for the pressure deviator.

We represent the non-equilibrium distribution function for the fourteen moments as

$$f = f_M (a + a_i C_i + a_{ij} C_i C_j + b_i C^2 C_i + b C^4), \quad (170)$$

where the fourteen coefficients  $a, a_i, a_{ij}, b_i$  and  $b$  do not depend on the peculiar velocity.

At equilibrium the contracted fourth moment reads

$$p_{ijj}^{(0)} = \int m C^4 f_M d\mathbf{c} = 15\rho \left( \frac{kT}{m} \right)^2, \quad (171)$$

and it is convenient to introduce a dimensionless non-equilibrium part of  $p_{ijj}$ , denoted by  $\Delta$  and defined by

$$\Delta = \frac{1}{15\rho} \left( \frac{m}{kT} \right)^2 [p_{ijj} - p_{ijj}^{(0)}] = \frac{1}{15\rho} \left( \frac{m}{kT} \right)^2 \int m C^4 (f - f_M) d\mathbf{c}. \quad (172)$$

The fourteen coefficients  $a, a_i, a_{ij}, b_i, b$  are determined from the definition of the basic fields (156), (157) and (172) together with the non-equilibrium distribution function (170), yielding

$$f = f_M \left\{ 1 + \frac{2\beta^2}{\rho} p_{\langle ij \rangle} C_i C_j + \frac{8\beta^2}{5\rho} q_i C_i \left( \beta C^2 - \frac{5}{2} \right) + \left( \frac{15}{8} - \frac{5\beta C^2}{2} + \frac{\beta^2 C^4}{2} \right) \Delta \right\}. \quad (173)$$

We can now determine the constitutive equations for the moments of the distribution function by inserting (173) into the expressions (163) and subsequent integration of the resulting equations. Hence, it follows

$$p_{\langle ijk \rangle} = 0, \quad p_{ijjk} = 28 \frac{kT}{m} q_k, \quad q_{ij} = \frac{5}{2} \rho \left( \frac{kT}{m} \right)^2 [1 + \Delta] \delta_{ij} + \frac{7}{2} \frac{kT}{m} p_{\langle ij \rangle}. \quad (174)$$

The constitutive equations for the cooling rate  $\zeta$  and for the production terms  $P_{\langle ij \rangle}, Q_i, P$  are obtained in the same manner and it follows from (81), (164), (165) and (166) through integration that

$$\zeta = \frac{1}{3\tau} \sqrt{\frac{T}{T_0}} (1 - \alpha^2) \left[ 1 + \frac{3\Delta}{16} \right], \quad P_{\langle ij \rangle} = -\frac{1}{5\tau} \sqrt{\frac{T}{T_0}} (1 + \alpha) (3 - \alpha) p_{\langle ij \rangle}, \quad (175)$$

$$Q_i = -\frac{1}{60\tau} \sqrt{\frac{T}{T_0}} (1 + \alpha) [49 - 33\alpha] q_i, \quad (176)$$

$$P = -\frac{\rho}{\tau} \left( \frac{kT}{m} \right)^2 (1 - \alpha^2) (2\alpha^2 + 9) \sqrt{\frac{T}{T_0}} \left\{ 1 + \frac{30\alpha^2(1 - \alpha) + 271 - 207\alpha}{(2\alpha^2 + 9)(1 - \alpha)} \frac{\Delta}{16} \right\}. \quad (177)$$

We have introduced in the above equations a mean free time in terms of a reference temperature  $T_0$ , namely,  $\tau = \sqrt{\frac{m}{\pi k T_0}} / 4nd^2$ . Furthermore, we have considered in the above expressions only linear terms in  $\Delta$ . For the elastic case  $\alpha = 1$  and the production terms (175) – (177) reduce to

$$\zeta = 0, \quad P_{\langle ij \rangle} = -\frac{4}{5\tau} \sqrt{\frac{T}{T_0}} p_{\langle ij \rangle}, \quad Q_i = -\frac{8}{15\tau} \sqrt{\frac{T}{T_0}} q_i, \quad P = -\frac{8\rho}{\tau} \left( \frac{kT}{m} \right)^2 \sqrt{\frac{T}{T_0}} \Delta. \quad (178)$$

From the knowledge of the constitutive equations in terms of the basic fields, we get – by inserting (174) – (177) into the balance equations (158), (159), (161), (162), (168)

and (169) – the following linearized system of fourteen field equations for  $\varrho$ ,  $v_i$ ,  $T$ ,  $p_{\langle ij \rangle}$ ,  $q_i$  and  $\Delta$ :

$$\mathcal{D}\varrho + \varrho \frac{\partial v_i}{\partial x_i} = 0, \quad \varrho \mathcal{D}v_i + \frac{\partial p}{\partial x_i} + \frac{\partial p_{\langle ij \rangle}}{\partial x_j} = 0, \quad (179)$$

$$\mathcal{D}T + \frac{2}{3nk} \left( \frac{\partial q_i}{\partial x_i} + p \frac{\partial v_i}{\partial x_i} \right) + \frac{T}{3\tau} \sqrt{\frac{T}{T_0}} (1 - \alpha^2) \left[ 1 + \frac{3\Delta}{16} \right] = 0, \quad (180)$$

$$\mathcal{D}p_{\langle ij \rangle} + \frac{4}{5} \frac{\partial q_{\langle i}}{\partial x_{j \rangle}} + 2p \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = -\frac{1}{5\tau} \sqrt{\frac{T}{T_0}} (1 + \alpha)(3 - \alpha) p_{\langle ij \rangle}, \quad (181)$$

$$\begin{aligned} \mathcal{D}q_i + \frac{kT}{m} \frac{\partial p_{\langle ij \rangle}}{\partial x_j} + \frac{5(kT)^2}{2m} \left( n \frac{\partial \Delta}{\partial x_i} + \Delta \frac{\partial n}{\partial x_i} \right) + \frac{5k^2 T n}{2m} \frac{\partial T}{\partial x_i} \\ = -\frac{1}{60\tau} \sqrt{\frac{T}{T_0}} (1 + \alpha) [49 - 33\alpha] q_i, \end{aligned} \quad (182)$$

$$\begin{aligned} 15\varrho \left( \frac{kT}{m} \right)^2 \mathcal{D}\Delta + \frac{8kT}{m} \frac{\partial q_j}{\partial x_j} = \frac{\varrho}{\tau} \left( \frac{kT}{m} \right)^2 (1 + \alpha) \sqrt{\frac{T}{T_0}} \left\{ (1 - 2\alpha^2)(1 - \alpha) \right. \\ \left. - [81 - 17\alpha + 30\alpha^2(1 - \alpha)] \frac{\Delta}{16} \right\}. \end{aligned} \quad (183)$$

In (183) we have used (179) and (180) in order to eliminate the time derivatives of  $\varrho$  and  $T$  and neglected products of  $\Delta$ .

Now we search for spatially homogeneous solutions of the fourteen field equations where the fields depend only on time. The field equations (179) – (183) in this case read

$$\frac{d\varrho}{dt_*} = 0, \quad \frac{dv_i}{dt_*} = 0, \quad (184)$$

$$\frac{dT_*}{dt_*} + \frac{T_*^{\frac{3}{2}}}{3} (1 - \alpha^2) \left[ 1 + \frac{3\Delta}{16} \right] = 0, \quad (185)$$

$$\frac{dp_{\langle ij \rangle}^*}{dt_*} = -\frac{(1 + \alpha)(3 - \alpha)}{5} \sqrt{T_*} p_{\langle ij \rangle}^*, \quad (186)$$

$$\frac{dq_i^*}{dt_*} = -\frac{(1 + \alpha)}{60} \sqrt{T_*} [49 - 33\alpha] q_i^*, \quad (187)$$

$$\frac{d\Delta}{dt_*} = \frac{1 + \alpha}{15} \sqrt{T_*} \left\{ (1 - 2\alpha^2)(1 - \alpha) - [81 - 17\alpha + 30\alpha^2(1 - \alpha)] \frac{\Delta}{16} \right\}. \quad (188)$$

Above, the following dimensionless quantities were introduced: time  $t_* = t/\tau$ , temperature  $T_* = T/T_0$ , pressure deviator  $p_{\langle ij \rangle}^*$  and heat flux vector  $q_i^*$ .

We may infer from (184) that the mass density and the velocity fields remain constant in time, while (185) – (188) compose a system of coupled differential equations for the determination of the temperature  $T_*$ , pressure deviator  $p_{\langle ij \rangle}^*$ , heat flux vector  $q_i^*$  and fourth moment  $\Delta$ . Since this system of equations is non-linear, it was solved numerically by considering the initial conditions  $T_*(0) = 1$ ,  $p_{\langle ij \rangle}^*(0) = 1$ ,  $q_i^*(0) = 1$  and  $\Delta(0) = 1$ . The left frame of Figure 3 shows the time decay of the temperature (solid line) in comparison with Haff's law (dashed line) which is the solution of (185) when  $\Delta = \text{constant}$  (see (190) below). We may conclude that the temperature decay  $T_*$  follows closely Haff's law and by

increasing the restitution coefficient the time decay of the temperature is less accentuated. Furthermore, from the right frame of the Figure 3 we may observe that the pressure deviator, the heat flux vector and the dimensionless fourth moment decay also with time and the trend to equilibrium is more pronounced for the pressure deviator followed by the heat flux vector and the dimensionless fourth moment. The curves in the right frame were obtained for the restitution coefficient  $\alpha = 0.75$ . By increasing the value of the restitution coefficient the time decay of the pressure deviator and dimensionless fourth moment are more accentuated but the one of the heat flux vector is less pronounced, since it is connected with the transport of energy which is directly affected by the inelasticity.

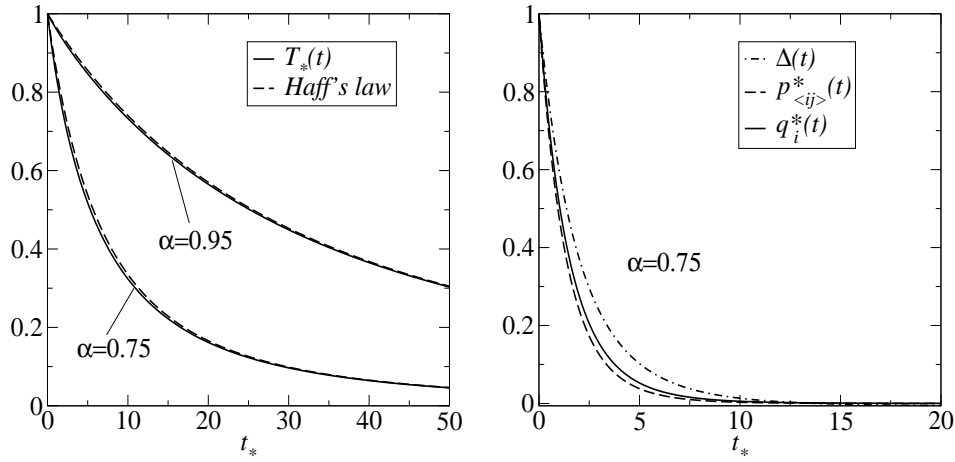


Figure 3: Left frame: time decay of the temperature. Right frame: time decay of the pressure deviator, heat flux vector and fourth moment.

From the analysis of the differential equations (186) we note that the pressure deviator and the heat flux vector do not evolve with respect to time when the initial conditions for these fields vanish. However, a vanishing initial condition for  $\Delta$  implies from (188) that it could evolve with time. An interesting case is the one where the fourth moment remains constant in time and which is expressed by the condition that the right-hand side of (188) must vanish, i.e.,

$$\Delta = \frac{16(1-\alpha)(1-2\alpha^2)}{30\alpha^2(1-\alpha) + 81 - 17\alpha} = a_2. \quad (189)$$

The above expression for  $\Delta$  is the same as that found for the coefficient  $a_2$  which follows from the Chapman-Enskog method (see (96)). We note that  $\Delta$  vanishes in the elastic case where  $\alpha = 1$ .

If we substitute the expression of  $\Delta$  given by (189) into (186) and then integrate the resulting equation, we get Haff's law

$$T_*(t) = \frac{1}{\left\{ 1 + \frac{(1-\alpha^2)}{6} \left[ 1 + \frac{3(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)} \right] t_* \right\}^2}. \quad (190)$$

By considering the elastic case  $\alpha = 1$ , the temperature  $T_*$  remains constant in time

and

$$p_{\langle ij \rangle}^*(t) = p_{\langle ij \rangle}^*(0)e^{-\frac{4}{5}\sqrt{T_*}t_*}, \quad q_i^*(t) = q_i^*(0)e^{-\frac{8}{15}\sqrt{T_*}t_*}, \quad (191)$$

i.e, both decay exponentially with time, which is a well known result.

In the case of a five field theory described by the fields of mass density, velocity and temperature, the pressure deviator and the heat flux vector are no longer fields but constitutive quantities. The constitutive equations for these quantities are obtained from the eight remaining field equations (181) and (182) by inserting the equilibrium values  $p_{\langle ij \rangle} = 0$  and  $q_i = 0$  on the left-hand side of these equations, yielding the laws of Navier-Stokes and Fourier

$$p_{\langle ij \rangle} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}, \quad q_i = -\lambda \frac{\partial T}{\partial x_i} - \vartheta \frac{\partial n}{\partial x_i}, \quad (192)$$

respectively. The transport coefficients of shear viscosity  $\mu$ , thermal conductivity  $\lambda$  and the coefficient  $\vartheta$  are given by

$$\mu = \frac{5}{4d^2} \sqrt{\frac{mkT}{\pi}} \frac{1}{(1+\alpha)(3-\alpha)}, \quad \lambda = \frac{75k}{2d^2m} \sqrt{\frac{mkT}{\pi}} \frac{1}{(1+\alpha)(49-33\alpha)}, \quad (193)$$

$$\vartheta = \frac{75kT}{2nd^2m} \sqrt{\frac{mkT}{\pi}} \frac{16(1-\alpha)(1-2\alpha^2)}{(1+\alpha)(49-33\alpha)[30\alpha^2(1-\alpha)+81-17\alpha]}. \quad (194)$$

We note that if we consider a thirteen moment theory the heat flux vector is proportional only to the temperature gradient and the dependence on the particle number gradient does not show up. This fact can be understood by observing that this dependence comes out from the underlined term in (182) which depends exclusively on the dimensionless fourth moment. Furthermore, the transport coefficients (193) and (194) are not the same as those obtained from the Chapman-Enskog method (104) and (105), and it seems that the equivalence of the two methods can be attained if we consider more moments of the distribution function in Grad's moment method.

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